Quantization of the Ramond-Neveu-Schwarz String

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Outline

Based on: Becker, Becker, Schwarz, String Theory and M-Theory, Chapter 4

A brief review of the Classical Theory

In the RNS formalism, we add D fermionic partners $\psi^\mu(\sigma^a)$ to the Polyakov action

$$S = -\frac{1}{2\pi}\int d^2\sigma\Big(\partial_a X^\mu \partial^a X_\mu + \bar{\psi}^\mu \rho^a \partial_a \psi_\mu\Big) \tag{1}$$

Here we take $\alpha' = 1/2$. The spinor fields $\psi^\mu(\sigma^a)$ are are two-component spinors on the world sheet and vectors under Lorentz transformations of the D-dimensional space-time.

ρ^a represent the two-dimensional Dirac matrices, which obey the Dirac algebra

$$\{\rho^a, \rho^b\} = 2\eta^{ab}, \qquad \rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

Classically, the fermionic world-sheet field ψ^μ is made of Grassmann numbers, which implies that it satisfies the anticommutation relations

$$\{\psi^\mu, \psi^\nu\} = 0 \tag{3}$$

The spinor ψ^μ has two components

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \tag{4}$$

We we define the Dirac conjugate of a spinor as

$$\bar{\psi}^\mu = \psi^\dagger \beta = i\big(\psi_+^{\mu *}, -\psi_-^{\mu *}\big) \tag{5}$$

Since the Dirac matrices are purely real, Eq. (2) is a Majorana representation, and the Majorana spinors ψ^μ are real

$$\psi_\pm^* = \psi_\pm \qquad \Longrightarrow \qquad \bar{\psi}^\mu = i\big(\psi_+^\mu, -\psi_-^\mu\big) \tag{6}$$

In this notation the fermionic part of the action is (suppressing the Lorentz index)

$$S_f = \frac{i}{\pi}\int d\tau d\sigma \Big(\psi_- \cdot \partial_+ \psi_- + \psi_+ \cdot \partial_- \psi_+\Big) \tag{7}$$

$$= \frac{i}{2\pi}\int d\sigma^+ d\sigma^- \Big(\psi_- \cdot \partial_+ \psi_- + \psi_+ \cdot \partial_- \psi_+\Big) \tag{8}$$

The equation of motion for the two spinor components is the Dirac equation, which now takes the form

$$\frac{\delta S_f}{\delta \psi_\pm} = 0 \qquad \Longrightarrow \qquad \partial_+ \psi_- = 0 \quad \text{and} \quad \partial_- \psi_+ = 0 \tag{9}$$

These equations describe left-moving and right-moving waves. For spinors in two dimensions, these are the Weyl conditions. Thus the fields $\psi_\pm$ are Majorana-Weyl spinors.

Global world-sheet supersymmetry

The RNS action is invariant under the infinitesimal transformations

$$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu, \tag{10}$$

$$\delta_\epsilon \psi^\mu = \rho^a \partial_a X^\mu \epsilon \tag{11}$$

where ϵ is a constant infinitesimal Majorana spinor that consists of anticommuting Grassmann numbers.

$$\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix} \tag{12}$$

In this convention the SUSY transformations can be written as

$$\begin{aligned} \delta_\epsilon X^\mu &= i\big(\epsilon_+\psi_-^\mu - \epsilon_-\psi_+^\mu\big), \\ \delta_\epsilon \psi_-^\mu &= -2\partial_- X^\mu \epsilon_+, \\ \delta_\epsilon \psi_+^\mu &= 2\partial_+ X^\mu \epsilon_-. \end{aligned} \tag{13}$$

In world-sheet light-cone coordinates, the action of the RNS string can be written

$$S = \frac{1}{2\pi}\int d\sigma^+ d\sigma^- \Big(2\,\partial_+ X\cdot\partial_- X + i\psi_-\cdot\partial_+\psi_- + i\psi_+\cdot\partial_-\psi_+\Big) \tag{14}$$

We may verify the above action is invariant under SUSY transformation (13).

Supercurrent and the Super-Virasoro Constraints

These two currents are the supercurrent, which arises from the supersymmetry of the action, and the stress-energy tensor, which arises from the translational symmetry of the action.

Since the SUSY is a global worldsheet symmetry we get, by Noether's theorem, an associated conserved current, called the worldsheet supercurrent. Taking the supersymmetry spinor parameter ϵ to be worldsheet coordinate dependent, the action varies, under this local SUSY, as

$$\delta S \sim \int d^2\sigma(\partial_a\bar{\epsilon})J^a, \qquad J^a_A = -\frac{1}{2}\left(\rho^b\rho^a\psi_\mu\right)_A\partial_b X^\mu \tag{15}$$

The supercurrent J^a_A are conversed:

$$(\rho_a)_{AB}J^a_B = 0 \tag{16}$$

In fact, J^a_A has only two independent components, which can be denoted J_+ and J_-. In terms of world-sheet light-cone coordinates,

$$J_+ = \psi^\mu_+\partial_+X_\mu, \qquad \text{and} \qquad J_- = \psi^\mu_-\partial_-X_\mu \tag{17}$$

$$\partial_-J_+ = \partial_+J_- = 0 \tag{18}$$

The next current of our theory is the current correspoding to translational symmetry of the RNS action. This current is called the stress-energy tensor and it is given by

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu + \frac{1}{4}\bar{\psi}^\mu \rho_a \partial_b \psi_\mu + \frac{1}{4}\bar{\psi}^\mu \rho_b \partial_a \psi_\mu - (\text{trace}) \tag{19}$$

In terms of world-sheet light-cone coordinates,

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\mu} \tag{20}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\mu} \tag{21}$$

$$T_{+-} = T_{-+} = 0 \tag{22}$$

The energy-momentum tensor satisfies

$$\partial_- T_{++} = \partial_+ T_{--} = 0 \tag{23}$$

The problem of negative-norm states appears also in the supersymmetric theory. In the supersymmetric RNS theory, we must impose super-Virasoro conditions:

$$J_+ = J_- = T_{++} = T_{--} = 0 \tag{24}$$

Boundary conditions and mode expansions

The possible boundary conditions and mode expansions for the bosonic fields X^μ are exactly the same as for the case of the bosonic string theory in the chapter 2.

For closed string we should impose the **periodic condition**

$$X^\mu(\sigma + \pi, \tau) \;=\; X^\mu(\sigma, \tau) \tag{25}$$

For open string, we have two boundary conditions:

- The first boundary condition is called the **Numann boundary condition**:

$$\partial_\sigma X^\mu = 0 \quad \text{at } \sigma = 0,\, \pi \tag{26}$$

  In this case the component of the momentum normal to the boundary of the world sheet vanishes. Physically, they mean that no momentum is flowing through the ends of the string.

- The second boundary condition is called the **Dirichlet boundary condition**:

$$X^\mu\Big|_{\sigma=0} = X_0^\mu, \qquad \text{and} \quad X^\mu\Big|_{\sigma=\pi} = X_\pi^\mu \tag{27}$$

  where X_0^μ and X_π^μ are constants. In this case the positions of the two string ends are fixed. The modern interpretation is that X_0^μ and X_π^μ represent the positions of Dp-branes.

The general solution of the wave equation of the closed string is given by

$$X^\mu(\tau,\sigma) = X^\mu_{\rm R}(\tau-\sigma) + X^\mu_{\rm L}(\tau+\sigma) \tag{28}$$

$$X^\mu_{\rm R}(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}\,e^{-2in(\tau-\sigma)}, \tag{29}$$

$$X^\mu_{\rm L}(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}\,e^{-2in(\tau+\sigma)} \tag{30}$$

where x^μ is a **center-of-mass position** and p^μ is the **total string momentum**, describing the free motion of the string center of mass. The exponential terms represent the string excitation modes. Notice that the positive and negative modes are conjugate to each other

$$\alpha_{-n} = (\alpha_n)^* \quad \text{and} \quad \tilde{\alpha}_{-n} = (\tilde{\alpha}_n)^* \tag{31}$$

Open-string mode expansion

For Numann boundary codition, the mode expansion for the open string is

$$X^\mu(\tau,\sigma) = x^\mu + p^\mu\,\tau + i\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\,e^{-in\tau}\,\cos(n\sigma) \tag{32}$$

$$= x^\mu + \alpha^\mu_0\,\tau + i\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\,e^{-in\tau}\,\cos(n\sigma) \tag{33}$$

Here $\alpha^\mu_0 = \sqrt{2\alpha'}\,p^\mu = p^\mu$, since we take $\ell_s^2 = 2\alpha' = 1$.

Next we consider the Fermionic part.

$$S_f \sim \int d^2\sigma \Big(\psi_- \partial_+ \psi_- \psi_+ \partial_- \psi_+ \Big) \tag{34}$$

By considering variations of the fields $\psi_\pm$ one finds that the action is stationary if the equations of motion are satisfied. The boundary terms in the variation of the action,

$$0 = \delta S_f \sim \int d\sigma^0 \big(\psi_+ \delta\psi_+ - \psi_- \delta\psi_- \big)\Big|_{\sigma=0}^{\sigma=\pi} \tag{35}$$

There are several ways to achieve this.

Open Strings

In the case of open strings the two terms in (35) must vanish separately.

$$\psi_+ = \pm\psi_- \tag{36}$$

The overall relative sign between ψ_+ and ψ_- is a matter of convention. Here we set

$$\psi_+^\mu\Big|_{\sigma=0} = \psi_-^\mu\Big|_{\sigma=0} \tag{37}$$

The relative sign at the end $\sigma = \pi$ then becomes meaningful, and there are two possible cases:

- Ramond boundary condition: $\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi}$
- Neveu-Schwarz boundary condition: $\psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi}$

Boundary conditions and mode expansions for open strings:

- Ramond boundary condition:

$$\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi} \tag{38}$$

The mode expansion of the fermionic field in the R (Ramond) sector takes the form

$$\psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu\, e^{-in(\tau-\sigma)}, \tag{39}$$

$$\psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu\, e^{-in(\tau+\sigma)} \tag{40}$$

The Majorana condition requires these expansions to be real, and hence

$$\psi_\pm^* = \psi_\pm \qquad \Longrightarrow \qquad d_{-n}^\mu = d_n^{\mu\dagger} \tag{41}$$

- Neveu-Schwarz boundary condition:

$$\psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi} \tag{42}$$

The mode expansion in the NS (Neveu-Schwarz) sector is

$$\psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu\, e^{-in(\tau-\sigma)}, \tag{43}$$

$$\psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu\, e^{-in(\tau+\sigma)} \tag{44}$$

Closed Strings

Closed-string boundary conditions give two sets of fermionic modes, corresponding to the left- and right-moving sectors. There are two possible periodicity conditions

$$\psi_\pm(\sigma) = \pm\psi_\pm(\sigma+\pi) \tag{45}$$

It is possible to impose the periodicity (R) or antiperiodicity (NS) of the right- and left-movers separately. For the right-movers, the mode expansion of the fermionic field takes

$$\textbf{R:}\quad \psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu\, e^{-2in(\tau-\sigma)}, \tag{46}$$

$$\textbf{NS:}\quad \psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu\, e^{-2in(\tau-\sigma)} \tag{47}$$

while for the left-movers we can take

$$\textbf{R:}\quad \psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu\, e^{-2in(\tau+\sigma)}, \tag{48}$$

$$\textbf{NS:}\quad \psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu\, e^{-2in(\tau+\sigma)} \tag{49}$$

Corresponding to the different pairings of the left- and right-movers there are four distinct closed-string sectors. States in the NS-NS and R-R sectors are space-time bosons, while states in the NS-R and R-NS sectors are spacetime fermions.

Canonical quantization of the RNS string

The modes in the Fourier expansion of the space-time coordinates satisfy the same commutation relations as in the case of the bosonic string, namely

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \tag{50}$$

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{51}$$

and the rest commutation relations are 0.

The fermionic fields ψ^μ satisfy the canonical anticommutation relations,

$$\{\psi_A^\mu(\tau,\sigma), \psi_B^\nu(\tau,\sigma')\} = \pi\eta^{\mu\nu}\delta_{AB}\,\delta(\sigma-\sigma') \tag{52}$$

This imply that the Fourier coeffcients satisfy

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}, \qquad \{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0} \tag{53}$$

Since the space-time metric $\eta^{00} = -1$, the time components of the fermionic modes give rise to negative-norm states. For example,

$$d_{-1}^0|0\rangle_{\mathrm{R}}: \ \texttt{Norm} = {}_{\mathrm{R}}\langle 0|d_1^0 d_{-1}^0|0\rangle_{\mathrm{R}} = -1, \quad b_{-1/2}^0|0\rangle_{\mathrm{NS}}: \ \texttt{Norm} = {}_{\mathrm{NS}}\langle 0|b_{1/2}^0 b_{-1/2}^0|0\rangle_{\mathrm{NS}} = -1 \tag{54}$$

These negative-norm states are decoupled as a consequence of the superconformal symmetry of the RNS string.

The Fock space ground state in the two sectors is defined by

$$\textbf{NS Sector:} \quad \alpha_n^\mu |0\rangle_{\rm NS} = b_r^\mu |0\rangle_{\rm NS} = 0 \quad \text{for} \quad n, r > 0 \tag{55}$$

$$\textbf{R Sector:} \quad \alpha_n^\mu |0\rangle_{\rm R} = d_n^\mu |0\rangle_{\rm R} = 0 \quad \text{for} \quad n > 0 \tag{56}$$

Excited states are constructed by acting with the negative modes (or raising modes) of the oscillators. Acting with the negative modes increases the mass of the states.

In the NS sector there is a unique ground state, which corresponds to a state of spin 0 in space-time. Since all the oscillators transform as space-time vectors, the excited states that are obtained by acting with raising operators are also space-time bosons.

By contrast, in the R sector the ground state is degenerate. The operators d_0^μ can act without changing the mass of a state, because they commute with the number operator N, defined below, whose eigenvalue determines the mass squared. These zero modes satisfy the algebra

$$\left\{d_0^\mu, d_0^\nu\right\} = \eta^{\mu\nu} \tag{57}$$

Aside from a factor of two, this is identical to the Dirac algebra

$$\Gamma^\mu = \sqrt{2}\, d_0^\mu, \qquad \left\{\Gamma^\mu, \Gamma^\nu\right\} = 2\eta^{\mu\nu} \tag{58}$$

As a result, the ground states in the R sector must furnish a representation of the D-dimensional Dirac algebra. Further, the states at each mass level must furnish a representation of this algebra. The ground states, in fact, should be an irreducible representation of the algebra (58). The irreducible representation correspond to spinors of $\mathrm{SO}(1,9)$. Therefore the R-sector ground state is a space-time fermion. Since all of the oscillators (α_n^μ and d_n^μ) are space-time vectors, and every state in the R sector can be obtained by acting with raising operators on the R-sector ground state, all R-sector states are space-time fermions.

As in the bosonic string, the densities of momentum and angular momentum along the string can be described as Noether currents associated with the global Poincaré symmetry

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu \tag{59}$$

$$\psi^\mu \to \Lambda^\mu{}_\nu \psi^\nu \tag{60}$$

The Noether currents are:

$$\mathcal{P}_a^\mu = \frac{1}{\pi}\partial_a X^\mu \tag{61}$$

$$\mathcal{J}_a^{\mu\nu} = \frac{1}{\pi}\Big(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu + \bar{\psi}^\mu \rho_a \psi^\nu\Big) \tag{62}$$

The conserved Charges are given by

$$p^{\mu} = \int_0^{\pi} d\sigma \mathcal{P}_0^{\mu} = \frac{1}{\pi}\int_0^{\pi} d\sigma \dot{X}^{\mu}(\tau, \sigma) \tag{63}$$

$$J^{\mu\nu} = \int_0^{\pi} d\sigma \mathcal{J}_0^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + K^{\mu\nu} \tag{64}$$

where

$$l^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu} \tag{65}$$

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^{\mu}\alpha_n^{\nu} - \alpha_{-n}^{\nu}\alpha_n^{\mu}\right) \tag{66}$$

are from the bosonic part while the contribution from fermionic part is:

NS: $$K^{\mu\nu} = -i\sum_{r=1/2}^{\infty}\left(b_{-r}^{\mu}b_r^{\nu} - b_{-r}^{\nu}b_r^{\mu}\right) \tag{67}$$

R: $$K^{\mu\nu} = -\frac{i}{2}\left[d_0^{\mu},\, d_0^{\nu}\right] - i\sum_{n=1}^{\infty}\left(d_{-n}^{\mu}d_n^{\nu} - d_{-n}^{\nu}d_n^{\mu}\right) \tag{68}$$

Using the canonical (anti-)commutation relations, we may verify the Poincaré algebra

$$[p^{\mu},\, p^{\nu}] = 0 \tag{69}$$

$$[p^{\mu},\, J^{\nu\rho}] = -i\eta^{\mu\nu}p^{\rho} + i\eta^{\mu\rho}p^{\nu} \tag{70}$$

$$[J^{\mu\nu},\, J^{\sigma\rho}] = i\eta^{\mu\sigma}J^{\nu\rho} + i\eta^{\nu\rho}J^{\mu\sigma} - i\eta^{\mu\rho}J^{\nu\sigma} - i\eta^{\nu\sigma}J^{\nu\rho} \tag{71}$$

Super-Virasoro algebra

The super-Virasoro algebra is spanned by the modes of T_{ab} and J_a. For the open string

$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma} T_{++} = L_m^{(b)} + L_m^{(f)} \tag{72}$$

- The contribution coming from the bosonic modes is

$$L_m^{(b)} = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha_{-n}\cdot\alpha_{m+n} : \qquad m \in \mathbb{Z} \tag{73}$$

- The contribution of the fermionic modes in the NS sector is

$$L_m^{(f)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2}\left(r+\frac{m}{2}\right) : b_{-r}\cdot b_{m+r} : \qquad m \in \mathbb{Z} \tag{74}$$

The modes of the supercurrent in the NS sector are

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{r+n} \qquad r \in \mathbb{Z}+1/2 \tag{75}$$

The operators L_0 can be written in the form

$$L_0 = \frac{1}{2}\alpha_0^2 + N \tag{76}$$

where the number operator N is given by

$$N = N^\alpha + N^b, \qquad N^\alpha = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n, \quad N^b = \sum_{r=1/2}^{\infty} r\, b_{-r}\cdot b_r \tag{77}$$

- In the R sector

$$L_m^{(f)} \;=\; \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(n+\frac{m}{2}\right) : d_{-n}\cdot d_{m+n} : \qquad m\in\mathbb{Z} \tag{78}$$

while the modes of the supercurrent are

$$F_m \;=\; \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma} J_+ \;=\; \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_{m+n} \qquad m\in\mathbb{Z} \tag{79}$$

Note that there is no normal-ordering ambiguity in the definition of F_0.

In the NS sector the super-Virasoro algebra is

$$[L_m,\, L_n] \;=\; (m-n)L_{m+n} \;+\; \frac{D}{8}m(m^2-1)\,\delta_{m+n,0}, \tag{80}$$

$$[L_m,\, G_r] \;=\; \left(\tfrac{m}{2}-r\right) G_{m+r}, \tag{81}$$

$$\{G_r,\, G_s\} \;=\; 2L_{r+s} \;+\; \frac{D}{2}\left(r^2-\tfrac{1}{4}\right)\,\delta_{r+s,0} \tag{82}$$

Notice that the five generators L_0, $L_{\pm 1}$, $G_{\pm 1/2}$ form a closed subalgebra, without anomaly

$$\left[L_{\pm 1},\, L_0\right] \;=\; \pm L_{\pm 1}, \quad \left[L_1,\, L_{-1}\right] \;=\; L_0 \tag{83}$$

$$[L_0,\, G_{\pm 1/2}] \;=\; \mp G_{\pm 1/2}, \quad [L_{\pm 1},\, G_{\mp 1/2}] \;=\; \pm G_{\pm 1/2}, \quad [L_{\pm 1},\, G_{\pm 1/2}] \;=\; 0 \tag{84}$$

$$\{G_{1/2},\, G_{-1/2}\} \;=\; 2L_0, \quad \{G_{\pm 1/2},\, G_{\pm 1/2}\} \;=\; 2L_{\pm 1} \tag{85}$$

This algebra is known as $OSp(1|2)$.

In the R sector super-Virasoro algebra is

$$[L_m,\, L_n] \;=\; (m-n)L_{m+n} \;+\; \frac{D}{8}m^3\,\delta_{m+n,0}, \tag{86}$$

$$[L_m,\, F_n] \;=\; \left(\tfrac{m}{2}-n\right)F_{m+n}, \tag{87}$$

$$\{F_m,\, F_n\} \;=\; 2L_{m+n} \;+\; \frac{D}{2}m^2\,\delta_{m+n,0} \tag{88}$$

From the last line, we have

$$\{F_0,\, F_0\} \;=\; 2L_0 \qquad \Longrightarrow \qquad F_0^2 \;=\; L_0 \tag{89}$$

Physical state conditions

When quantizing the RNS string one can only require that the positive modes of the Virasoro generators annihilate the physical state. So in the NS sector the physical state conditions are

$$G_r\,|\phi\rangle = 0, \quad r > 0 \tag{90}$$

$$L_m\,|\phi\rangle = 0, \quad m > 0 \tag{91}$$

$$\left(L_0 - a_{\mathrm{NS}}\right)|\phi\rangle = 0. \tag{92}$$

Here $|\phi\rangle = |\texttt{physics}\rangle$ donotes the physical state. The last of these conditions implies that $\alpha' M^2 = N - a_{\mathrm{NS}}$. Similarly, in the R sector the physical-state conditions are

$$F_n\,|\phi\rangle = 0, \quad n \geqslant 0 \tag{93}$$

$$L_m\,|\phi\rangle = 0, \quad m > 0 \tag{94}$$

$$\left(L_0 - a_{\mathrm{R}}\right)|\phi\rangle = 0. \tag{95}$$

In the above formulas a_{NS}, a_{R} and dimension D in the super-Virasoro algebra must be determined. Using superalgebra $\{F_0,\, F_0\} = 2L_0 \Longrightarrow F_0^2 = L_0$

$$0 = \left(L_0 - a_{\mathrm{R}}\right)|\phi\rangle = \left(F_0^2 - a_{\mathrm{R}}\right)|\phi\rangle = -a_{\mathrm{R}}\,|\phi\rangle \quad \Longrightarrow \quad a_{\mathrm{R}} = 0 \tag{96}$$

Since

$$
\begin{aligned}
F_0 &= \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_n \\
&= \alpha_0\cdot d_0 + \sum_{n=1}^{\infty}\Big(\alpha_{-n}\cdot d_n + d_{-n}\cdot\alpha_n\Big) \\
&= \frac{\ell_s}{\sqrt{8}}\left(\frac{2\alpha_0}{\ell_s}\cdot\sqrt{2}\,d_0 + \frac{\sqrt{8}}{\ell_s}\sum_{n=1}^{\infty}\left(\alpha_{-n}\cdot d_n + d_{-n}\cdot\alpha_n\right)\right) \\
&= \frac{\ell_s}{\sqrt{8}}\left(p\cdot\Gamma + \frac{\sqrt{8}}{\ell_s}\sum_{n=1}^{\infty}\left(\alpha_{-n}\cdot d_n + d_{-n}\cdot\alpha_n\right)\right),
\end{aligned}
\tag{97}
$$

the F_0 equation $F_0\,|\phi\rangle = 0$ also can be written in the form

$$
\left(p\cdot\Gamma + \frac{2\sqrt{2}}{\ell_s}\sum_{n=1}^{\infty}\left(\alpha_{-n}\cdot d_n + d_{-n}\cdot\alpha_n\right)\right)|\phi\rangle = 0 \tag{98}
$$

This is a stringy generalization of the Dirac equation, known as the Dirac-Ramond equation.

Absence of negative-norm states

As in the discussion of the bosonic string in Chapter 2, there are specific values of a and D for which additional zero-norm states appear in the spectrum.

$$D = 10, \qquad a_{\mathrm{R}} = 0, \quad a_{\mathrm{NS}} = \frac{1}{2} \tag{99}$$

Let us consider a few simple examples of zero-norm spurious states. Recall that these are states that are orthogonal to physical states and decouple from the theory even though they satisfy the physical state conditions.

Example 1: Consider NS-sector states of the form

$$|\psi\rangle \;=\; G_{-1/2}\,|\chi\rangle \tag{100}$$

with $|\chi\rangle$ satisfying the conditions

$$G_{1/2}\,|\chi\rangle \;=\; G_{3/2}\,|\chi\rangle \;=\; \big(L_0 + \tfrac{1}{2} - a_{\mathrm{NS}}\big)\,|\chi\rangle \;=\; 0 \tag{101}$$

The last of these conditions is equivalent to $(L_0 - a_{\mathrm{NS}})\,|\psi\rangle = 0$.

$$\begin{aligned}[L_0,\, G_{-1/2}] = \frac{1}{2}G_{-1/2} \quad \Longrightarrow \quad (L_0 - a_{\mathrm{NS}})\,|\psi\rangle \;&=\; (L_0 - a_{\mathrm{NS}})G_{-1/2}\,|\chi\rangle \\ &=\; (L_0 - a_{\mathrm{NS}})G_{-1/2}\,|\chi\rangle \\ &=\; G_{-1/2}\big(L_0 + \tfrac{1}{2} - a_{\mathrm{NS}}\big)\,|\chi\rangle \end{aligned} \tag{102}$$

Consider $G_{1/2}\,|\chi\rangle = 0$

$$G_{1/2}\,|\psi\rangle \;=\; G_{1/2}G_{-1/2}\,|\chi\rangle \;=\; \big(-G_{-1/2}G_{1/2}+2L_0\big)\,|\chi\rangle \;=\; 2L_0\,|\chi\rangle \;=\; \big(2a_{\rm NS}-1\big)\,|\chi\rangle \tag{103}$$

We need $a_{\rm NS} = \dfrac{1}{2}$ and this is consistent with the condition $G_{3/2}\,|\chi\rangle = 0$. This choice gives a family of zero-norm spurious states $|\psi\rangle$.

$$\langle\psi|\psi\rangle \;=\; \langle\chi|G_{1/2}G_{-1/2}|\chi\rangle \;=\; 2\,\langle\chi|L_0|\chi\rangle \;=\; 0 \tag{104}$$

Moreover, the sate $|\psi\rangle$ is orthogonal to all physical states,

$$\langle\texttt{physics}\,|\,\psi\rangle \;=\; \big\langle\texttt{physics}\,\big|\,G_{-1/2}\,\big|\,\chi\big\rangle \;=\; \big\langle\chi\,\big|\,G_{1/2}\,\big|\,\texttt{physics}\big\rangle^{*} \;=\; 0 \tag{105}$$

Therefore, for $a_{\rm NS} = 1/2$ these are zero-norm spurious states.

Example 2: Now let us construct a second class of NS-sector zero-norm spurious states. Consider states of the form

$$|\psi\rangle \;=\; \big(G_{-3/2}+\lambda G_{-1/2}L_{-1}\big)\,|\chi\rangle \tag{106}$$

Suppose further that the state $|\chi\rangle$ satisfies

$$G_{1/2}\,|\chi\rangle \;=\; G_{3/2}\,|\chi\rangle \;=\; \big(L_0+1\big)\,|\chi\rangle \;=\; 0 \tag{107}$$

Similarly, the last of these conditions is equivalent to $(L_0-\frac{1}{2})\,|\psi\rangle = 0$.

Using the super-Virasoro algebra, we have

$$\begin{aligned}
G_{1/2}\,|\psi\rangle &= G_{1/2}\big(G_{-3/2} + \lambda G_{-1/2}L_{-1}\big)\,|\chi\rangle \\
&= \Big(2L_{-1} + \lambda\big(2L_0 - G_{-1/2}G_{1/2}\big)L_{-1}\Big)\,|\chi\rangle \\
&= \Big(2L_{-1} + 2\lambda L_{-1}\big(1 + L_0\big) - \lambda G_{-1/2}G_{-1/2}\Big)\,|\chi\rangle \\
&= \Big(2L_{-1} + 2\lambda L_{-1}\big(1 + L_0\big) - \lambda L_{-1}\Big)\,|\chi\rangle \\
&= \big(2 - \lambda\big)L_{-1}\,|\chi\rangle
\end{aligned} \tag{108}$$

and

$$\begin{aligned}
G_{3/2}\,|\psi\rangle &= G_{3/2}\big(G_{-3/2} + \lambda G_{-1/2}L_{-1}\big)\,|\chi\rangle \\
&= \Big(2L_0 + D + \lambda\big(2L_1 - G_{-1/2}G_{3/2}\big)L_{-1}\Big)\,|\chi\rangle \\
&= \Big(-2 + D + 2\lambda\big(2L_0 + L_{-1}L_1\big) - 2\lambda G_{-1/2}G_{1/2}\Big)\,|\chi\rangle \\
&= \big(-2 + D - 4\lambda\big)\,|\chi\rangle
\end{aligned} \tag{109}$$

Therefore, by the same reasoning as in the previous example, one concludes that $|\psi\rangle$ is a zeronorm spurious state if $\lambda = 2$ and $D = 10$.

Example 3: In the R-sector, such a set of zero-norm states can be built from R-sector states of the form

$$|\psi\rangle \;=\; F_0 F_{-1}\,|\chi\rangle \tag{110}$$

which satifies

$$F_1\,|\chi\rangle \;=\; (L_0+1)\,|\chi\rangle \;=\; 0 \tag{111}$$

This state satisfies $F_0\,|\psi\rangle = 0$. If it is also annihilated by L_1, then it is a physical state with zero-norm.

$$\begin{aligned} L_1\,|\psi\rangle \;=\; L_1 F_0 F_{-1}\,|\chi\rangle \;&=\; \left(\tfrac{1}{2}F_1 + F_0 L_1\right) F_{-1}\,|\chi\rangle \\ &=\; \Big(\tfrac{1}{2}\big(2L_0 + \tfrac{D}{2}\big) + F_0\big(\tfrac{3}{2}F_0 + F_{-1}L_1\big)\Big)\,|\chi\rangle \\ &=\; \frac{1}{4}\big(D-10\big)\,|\chi\rangle \end{aligned} \tag{112}$$

This vanishes for $D=10$ giving us another family of zero-norm spurious states for this space-time dimension.

Light-cone gauge quantization of the RNS

As in the case of the bosonic string, after gauge fixing there is a residual conformal symmetry that can be used to impose the light-cone gauge condition

$$X^+ = x^+ + p^+\tau \tag{113}$$

This is true for the RNS string as well. There is now also the freedom of applying local SUSY transformation that preserve the gauge choices. They turn out to be just sufficient to gauge away ψ^+ completely so that we may make the gauge choice

$$\psi^+ = 0 \tag{114}$$

In the R sector one should keep the zero mode d_0, which is a Dirac matrix.

Because of the super-Virasoro constraint, the X^- and ψ^- are not an independent degree of freedom in the light-cone gauge. Recall the super-Virasoro constraints:

$$J_+ = \psi_+ \cdot \partial_+ X = 0 \tag{115}$$

$$J_- = \psi_- \cdot \partial_- X\,, = 0 \tag{116}$$

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\mu} = 0 \tag{117}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\mu} = 0 \tag{118}$$

Let us concentrate on the open string in the NS sector. Consider the constraint $J_+ = 0$

$$0 \;=\; \psi_+ \cdot \partial_+ X \;=\; -\psi_+^+ \partial_+ X^- - \psi_+^- \partial_+ X^+ \;+\; \sum_{i=1}^{D-2} \psi_+^i \partial_+ X^i \tag{119}$$

$\Longrightarrow$

$$\psi_+^- \;=\; \left(\partial_+ X^+\right)^{-1} \sum_{i=1}^{D-2} \psi_+^i \partial_+ X^i \;=\; \frac{2}{p^+} \sum_{i=1}^{D-2} \psi_+^i \partial_+ X^i \tag{120}$$

Notice that

$$\partial_+ X^\mu \;=\; \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu \, e^{-in(\tau+\sigma)} \tag{121}$$

$$\psi_+^\mu(\tau,\sigma) \;=\; \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z}+1/2} b_r^\mu \, e^{-ir(\tau+\sigma)}, \tag{122}$$

We have

$$\begin{aligned}
\psi_+^- \;&=\; \frac{2}{p^+} \sum_{i=1}^{D-2} \psi_+^i \partial_+ X^i \\
&=\; \frac{1}{\sqrt{2} p^+} \sum_{i=1}^{D-2} \sum_{r \in \mathbb{Z}+1/2} \sum_{n=-\infty}^{+\infty} b_r^i \alpha_n^i \, e^{-i(n+r)(\tau+\sigma)} \\
&=\; \frac{1}{\sqrt{2} p^+} \sum_{s \in \mathbb{Z}+1/2} \sum_{i=1}^{D-2} \sum_{r=-\infty}^{+\infty} b_s^i \alpha_{s-r}^i \, e^{-is(\tau+\sigma)}
\end{aligned} \tag{123}$$

In terms of Fourier modes

$$\psi_+^- = \frac{1}{\sqrt{2}} \sum_{s\in\mathbb{Z}+1/2} b_s^- \, e^{-is(\tau+\sigma)}, \qquad b_s^- = \frac{1}{p^+} \sum_{i=1}^{D-2} \sum_{r=-\infty}^{+\infty} b_s^i \alpha_{s-r}^i \tag{124}$$

In the open-string NS sector, consider the constraint $J_- = 0$

$$\begin{aligned} 0 = \psi_- \cdot \partial_- X &= -\psi_-^+ \partial_- X^- - \psi_-^- \partial_- X^+ + \sum_i \psi_-^i \partial_- X^i \\ &= -\frac{p^+}{2} \psi_-^- + \sum_i \psi_-^i \partial_- X^i \end{aligned} \tag{125}$$

$\Longrightarrow$

$$\psi_-^- = \frac{2}{p^+} \sum_{i=1}^{D-2} \psi_-^i \partial_- X^i \tag{126}$$

Using the similar method, we have

$$\partial_+ X^- = \frac{1}{p^+} \Big(\partial_+ X^i \partial_+ X^i + \frac{i}{2} \psi^i \partial_+ \psi^i \Big) \tag{127}$$

This will give

$$\alpha_n^- = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left[\sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : + \sum_{r=-\infty}^{\infty} \Big(r - \frac{n}{2} \Big) : b_{n-r}^i b_r^i : \right] - \frac{a \delta_{n,0}}{2p^+} \tag{128}$$

Therefore, all the independent physical excitations are obtained in light-cone gauge by acting on the ground states with the transverse raising modes of the bosonic and fermionic oscillators.

*Lorentz symmetry

In light-cone gauge, Lorentz symmetry is not manifest. In the bosonic open-string sector

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + K^{\mu\nu} \tag{129}$$

$$E^{+\mu} = K^{+\mu} = 0 \tag{130}$$

$$E^{i-} = -i \sum_{n=1}^{\infty} \frac{1}{n}\Big(\alpha^i_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^i_n\Big) \tag{131}$$

$$K^{i-} = \frac{1}{p^+} \sum_{n=-\infty}^{\infty} \sum_{j=1}^{D-2} K^{ij}_{-n}\alpha^i_n \tag{132}$$

where

$$K^{ij}_m = -\frac{i}{2} \sum_{r=-\infty}^{\infty} \left(b^i_{m-r}b^j_r - b^j_{m-r}b^i_r\right) \tag{133}$$

After a long calculation, we find the Lorentz generators satify the usual Lorentz algebra, expect for the case of $[J^{i+}, J^{i-}]$, which is supposed to vanish, i.e.,

$$[J^{i+}, J^{i-}] = 0 \tag{134}$$

We will find the algebra is closed only if $D = 10$ and $a_{\rm NS} = 1/2$. For details, see GSW [1], page 211-213.

Analysis of the spectrum

- **The NS sector**

$$L_0 \;=\; \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \sum_{r=1/2}^{\infty} r b_{-r}\cdot b_r \;=\; \alpha' p^2 \;+\; N^{\alpha} \;+\; N^{b} \tag{135}$$

Form this we can get the mass formula

$$\alpha' M^2 \;=\; N^{\alpha} \;+\; N^{b} - a_{\text{NS}} \;=\; \sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i \;+\; \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i - \frac{1}{2} \tag{136}$$

The first two states in this sector are as follows:

- **The ground state** is annihilated by the positive lowering modes,

$$\alpha_n^i\,|0;k\rangle_{\text{NS}} \;=\; b_r^i\,|0;k\rangle_{\text{NS}} \;=\; 0 \qquad \text{for} \quad n, r > 0 \tag{137}$$

and the $|0;k\rangle$ is the eigenstate of p^{μ}:

$$p^{\mu}\,|0;k\rangle_{\text{NS}} \;=\; k^{\mu}\,|0;k\rangle_{\text{NS}} \tag{138}$$

The ground state in the NS sector is a scalar in space-time and the mass is given by

$$\alpha' M^2 \;=\; -\frac{1}{2} \tag{139}$$

As a result, the ground state of the RNS string in the NS sector is once again a tachyon.

- **The First excited state in the NS sector**

$$b^i_{-1/2}\,|0;k\rangle_{\rm NS} \tag{140}$$

The mass is

$$N^\alpha = 0, \quad N^b = \frac{1}{2} \qquad \alpha' M^2 \; = \; \frac{1}{2} - a_{\rm NS} \tag{141}$$

Since this is in light-cone gauge, the index i labels the $D - 2 = 8$ transverse directions. Since the operator $b^i_{-1/2}$ is acting on a bosonic ground state that is a space-time scalar, the resulting state is a space-time vector.

As a general rule, Lorentz invariance implies that physical states form representations of $\mathrm{SO}(D-1)$ for massive states and $\mathrm{SO}(D-2)$ for massless states. Therefore, the vector state $b^i_{-1/2}\,|0;k\rangle$ is massless.

$$\frac{1}{2} - a_{\rm NS} = 0 \qquad \Longrightarrow \qquad a_{\rm NS} \; = \; \frac{1}{2} \tag{142}$$

- **The R sector**

$$L_0 \;=\; \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty} n\, d_{-n}\cdot d_n \;=\; \alpha' p^2 + N^\alpha + N^d \tag{143}$$

Form this we can get the mass formula

$$\alpha' M^2 \;=\; N^\alpha + N^d - a_{\mathrm{R}} \;=\; \sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i + \sum_{n=1}^{\infty} n\, d_{-n}^i d_n^i, \qquad a_{\mathrm{R}} = 0 \tag{144}$$

In this sector the states are as follows:

- **The ground state** is the solution of

$$\alpha_n^i\,|0;k\rangle_{\mathrm{R}} \;=\; d_n^i\,|0;k\rangle_{\mathrm{R}} \;=\; 0 \qquad \text{for} \quad n > 0 \tag{145}$$

as well as the massless Dirac equation. As was discussed above, the solution of these equations is not unique, since the zero modes satisfy the ten-dimensional Dirac algebra. Thus the solution to these constraints gives a $Spin(9,1)$ spinor. The ground state in the R sector is described by a 32-component spinor.

In ten dimensions spinors can be restricted by Majorana and Weyl conditions, each of which reduces the number of components by a factor of two thereby reducing the total to 16 real components. Using the massless Dirac equation we can eliminates half of the components again. Thus in the end, the minimal possibility for a Ramond ground state has eight physical degrees of freedom corresponding to an irreducible spinor of $Spin(8)$.

Zero-point energies

First of all, let us recall the case of the bosonic open-string.

$$\alpha' M^2 = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{+\infty}\alpha^i_{-n}\alpha^i_n = \sum_{i=1}^{D-2}\sum_{n=1}^{+\infty} :\alpha^i_{-n}\alpha^i_n: \ + \ \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=1}^{+\infty}\left[\alpha^i_n,\, \alpha^i_{-n}\right]$$
$$= \sum_{i=1}^{D-2}\sum_{n=1}^{+\infty}\Big(N^i_n + \frac{1}{2}\Big)n \qquad (146)$$

For the ground state, all $N^i_n = 0$ and

$$M_0^2 = \frac{D-2}{2\alpha'}\sum_{n=1}^{\infty} n \qquad (147)$$

The sum on the right-hand side is divergent and needs to be regularized. Define the ζ-function as a sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (148)$$

This converges for $\mathrm{Re}(s) > 1$ and has a pole at $s = 1$. It can be continued around the pole, and $\zeta(-1) = -\dfrac{1}{12}$. So this is the value we assign to the $\sum_{n=1}^{\infty} n = -\frac{1}{12}$, by analogy to dimensional regularization [3]. Therefore

$$M_0^2 = -\frac{D-2}{24\alpha'} = -\frac{1}{\alpha'} \qquad (149)$$

The NS sector of the RNS string has

$$
\begin{aligned}
\alpha' M^2 &= \frac{1}{2}\sum_{i=1}^{D-2}\left\{\sum_{n=-\infty}^{\infty}\alpha_{-n}^i\alpha_n^i + \sum_{r=-\infty}^{\infty} r\, b_{-r}^i b_r^i\right\} \\
&= \frac{1}{2}\sum_{i=1}^{D-2}\left\{\sum_{n=-\infty}^{\infty} :\alpha_{-n}^i\alpha_n^i: + \sum_{r=-\infty}^{\infty} r :b_{-r}^i b_r^i: \right\} + \frac{1}{2}\sum_{i=1}^{D-2}\left\{\sum_{n=1}^{\infty}\left[\alpha_n^i,\, \alpha_n^i\right] - \sum_{r=1/2}^{\infty} r\left\{b_r^i,\, b_{-r}^i\right\}\right\} \\
&= \frac{1}{2}\sum_{i=1}^{D-2}\left\{\sum_{n=-\infty}^{\infty} :\alpha_{-n}^i\alpha_n^i: + \sum_{r=-\infty}^{\infty} r :b_{-r}^i b_r^i: \right\} + \frac{1}{2}(D-2)\left(\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\right)
\end{aligned} \tag{150}
$$

For the ground state,

$$
M_0^2 = \frac{D-2}{2\alpha'}\left(\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\right) = \frac{D-2}{2\alpha'}\cdot\left(1+\frac{1}{2}\right)\cdot\sum_{n=1}^{\infty} n = -\frac{1}{2\alpha'} \tag{151}
$$

Here we have used

$$
\sum_{n=1}^{\infty} n = 1+2+3+4+\cdots = \left(1+3+5+\cdots\right) + 2\cdot\left(1+2+3+\cdots\right) \tag{152}
$$

$\Longrightarrow$

$$
1+2+3+4+\cdots = -\left(1+3+5+7+\cdots\right) \tag{153}
$$

$\Longrightarrow$

$$
\sum_{r=1/2}^{\infty} r = \frac{1}{2}+\frac{3}{2}+\frac{5}{2}+\frac{7}{2}+\cdots = \frac{1}{2}\left(1+3+5+7+\cdots\right) = -\frac{1}{2}\sum_{n=1}^{\infty} n \tag{154}
$$

The GSO projection

The RNS model, as described so far, is an inconsistent quantum theory unless further conditions are imposed. The spectrum must be truncated in a vary specific manner first proposed by Gliozzi, Scherk and Olive. By GSO projection, the RNS string theory turns into a consistent theory and leads to a supersymmetric theory in ten-dimensional space-time.

There are several arguments that suggest that a truncation of the spectrum is required.

- The theory has a tachyon, and we would like eliminate it.
- It is unnatural to have anticommuting operators ψ^μ that map bosons to bosons.
- The spectrum is not space-time supersymmetric.

♢ Tachyon

First when interactions are included, this theory might not have a stable vacuum. Second there is no fermion in the spectrum with the same mass as the tachyon.

♢ $\psi^\mu |\phi\rangle$

Let $|\phi\rangle$ be a bosonic state, then state

$$\psi^{\mu_1}(\sigma_1)\psi^{\mu_2}(\sigma_2)\cdots\psi^{\mu_n}(\sigma_n)\,|\phi\rangle \tag{155}$$

is always bosonic for any n, since the ψ^μ are all bosonic.

For even n, there is nothing peculiar. But for odd n, this state may gives us the felling that all is not well. We are tempted to discarding the states that have odd n and keeping those of even n. This can be stated formally by introducing a quantum called $(-1)^F$ under which the Fermi fields ψ^μ are odd and the Bose fields X^μ are even. We fix the sign by saying that the massless vector has $(-1)^F = +1$, then for the state

$$\psi^{\mu_1}(\sigma_1)\psi^{\mu_2}(\sigma_2)\cdots\psi^{\mu_n}(\sigma_n)\left|\phi\right\rangle, \quad (-1)^F = (-1)^n. \tag{156}$$

The GSO projection:

Keeping only states of even n amounts to keeping only states of $(-1)^F = +1$.

♢ SUSY in 10-dimension

As will be explained shortly, the closed-string spectrum contains a massless gravitino (or two) with spin 3/2 and therefore the interacting theory wouldn't be consistent without supersymmetry. In particular, this requires an equal number of physical bosonic and fermionic modes at each mass level.

The GSO projection gives a supersymmetric theory in ten-dimensional spacetime. We will see evidence for this here, and a complete proof in the next chapter.

Majorana-Weyl Condition

The massless open string states consist of a vector and a spinor:

$$\begin{array}{cc} b^i_{-1/2}\,|0;k\rangle_{\rm NS} & |\psi\rangle = |a;k\rangle\, u(a;k) \quad \text{satisfies} \quad F_0\,|\psi\rangle = 0 \\ 8 & 32\times 2/(2\times 2\times 2) = 8 \end{array} \tag{157}$$

where a is a spinor index and k is the momentum.

A necessary condition for supersymmetry of the theory is that this pair of states should form a SUSY multiplet.

A massless vector field A^μ in $D = 10$ has 10 components, but only 8 transverse components describe independent propagating modes. The SUSY requires that the massless spinor should also have 8 propagating modes.

Number of components of the spinor in different dimensions:

Spinor	$D=2$	$D=4$	$D=10$
Dirac spinor	2 complex	4 complex	32 complex
Weyl spinor	1 complex	2 complex	16 complex
Majorana spinor	2 real	4 real	32 real
Majorana-Weyl spinor	1 real	γ^5 is imaginary in Majorana rep.	16 real

Majorana-Weyl Condition

Consider Dirac equation

$$\Gamma^\mu \partial_\mu \psi = 0. \tag{158}$$

If it is possible to choose the Dirac matrices Γ^μ to be all real or all imaginary, then it makes sense to impose a condition that the spinor ψ should be real. A representation of Gamma matrices in which they all real or all imaginary is called a Majorana representation, and real spinors are called Majorana spinors.

A Majorana representation of the $D = 10$ Dirac algebra is

$$\Gamma^0 = i\sigma_2 \otimes \mathbf{1} \tag{159}$$

$$\Gamma^i = \sigma_1 \otimes \gamma^i \qquad i = 1, \dots, 8 \tag{160}$$

$$\Gamma^9 = \sigma_3 \otimes \mathbf{1} \tag{161}$$

where γ^i are the 16×16 matrices and satisfy

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \qquad i, j = 1, \dots, 8 \tag{162}$$

It is easy to verify

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{163}$$

For details, see GSW [1], appendix 5.B or [15] Basic Concepts of String Theory, sec. 8.5.

Majorana-Weyl Condition

Whenever D is even one can define a matrix, analogous to γ^5 in four dimnesions, that can be used to define a chirality of spinors. In D=10, introduce

$$\Gamma_{11} = \Gamma^0\Gamma^1 \cdots \Gamma^9. \tag{164}$$

This matrix satisfies

$$\{\Gamma_{11}, \Gamma^\mu\} = 0 \qquad \text{and} \qquad (\Gamma_{11})^2 = \mathbf{1} \tag{165}$$

A spinor that satisfy

$$\Gamma_{11}\psi = \pm\psi \tag{166}$$

are said to have positive or negative chirality. The chirality projection operators are

$$P_\pm = \frac{1 \pm \Gamma_{11}}{2} \tag{167}$$

A spinor with a definite chirality is called a Weyl spinor.

In the Majorana representation given above, with the ten Γ^μ real, it is clear that Γ_{11} is real. Therefore given a real spinor χ, the two pieces of definite chirality $P_\pm\chi$ are also real. This means that the Majorana and Weyl conditions are compatible

The GSO projection

In $D = 10$, the massless spinors must be simultaneously Majorana and Weyl to ensure the massless sector forms a SUSY multiplet. The Weyl condition means that the ground-state spinor is an eigenstate of Γ_{11}. The generalization of this condition to an arbitrary fermion mass level requires the operator

$$\bar{\Gamma} \;=\; \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d_{-n}\cdot d_n} \tag{168}$$

which has the property

$$\{\bar{\Gamma},\, d_n^{\mu}\} \;=\; 0 \qquad \Longleftarrow \qquad \Gamma_{11} \;=\; \Gamma^0\Gamma^1\cdots\Gamma^9, \quad \gamma^{\mu}\sim d_0^{\mu} \tag{169}$$

Since ψ^{μ} is linear in the d_n^{μ}, we have

$$\{\bar{\Gamma},\, \psi^{\mu}(\tau,\sigma)\} \;=\; 0 \tag{170}$$

The GSO condition is

$$\bar{\Gamma}\,|\psi\rangle \;=\; |\psi\rangle \tag{171}$$

for a physical fermion.

In the NS sector, we define the G-parity as

$$G \;=\; (-1)^{\sum_{r=1/2}^{\infty} b_{-r}\cdot b_r + 1} \tag{172}$$

The GSO condition is

$$G\,|\phi\rangle \;=\; |\phi\rangle \tag{173}$$

for a physical boson.

The GSO projection eliminates the open-string tachyon from the spectrum, since

$$G\,|0\rangle_{\rm NS} = -\,|0\rangle_{\rm NS}\,. \tag{174}$$

The first excited state,

$$|\phi\rangle \;=\; b^{i}_{-1/2}\,|0;k\rangle_{\rm NS} \qquad M^2 = 0 \tag{175}$$

Since $\sum_{r=1/2}^{\infty} b_{-r}\cdot b_r = 1$,

$$G\,|\phi\rangle \;=\; |\phi\rangle\,, \tag{176}$$

so this state survives the projection. After the GSO projection, this massless vector boson becomes new ground state of the NS sector. This matches nicely with the fact that the ground state in the fermionic sector is a massless spinor. This is a first indication that the spectrum could be space-time supersymmetric after performing the GSO projection. At this point the GSO projection may appear to be an ad hoc condition, but actually it is essential for consistency.

The first massive level of open strings

- **NS sector**

$$\alpha' M^2 = N - \frac{1}{2} \qquad N = N^\alpha + N^b = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=1/2}^{\infty} r b^i_{-r} b^i_r \tag{177}$$

♢ $N^\alpha = 1,\ N^b = 0$ or $N^\alpha = 0,\ N^b = 1$

$$\alpha' M^2 = \frac{1}{2} \qquad \alpha^i_{-1}|0\rangle \quad \text{or} \quad b^i_{-1/2} b^j_{-1/2}|0\rangle \tag{178}$$

Obviously, the GSO projection will eliminate these states from the spectrum since they have minus G-parity.

♢ $N^\alpha = 1,\ N^b = 1/2$

$$\alpha' M^2 = 1 \qquad \underset{64}{\alpha^i_{-1} b^j_{-1/2}|0\rangle} \qquad \underset{56}{b^i_{-1/2} b^j_{-1/2} b^k_{-1/2}|0\rangle} \qquad \underset{8}{b^i_{-3/2}|0\rangle} \tag{179}$$

Obviously, the G-parity $(-1)^F = +1$ so these states survive the projection. Therefore this is the first massive level. These massive states combine into $\mathrm{SO}(9)$ representations:

$$\mathbf{128} = \mathbf{44} \oplus \mathbf{84} \tag{180}$$

- **R sector**

$$\alpha' M^2 = N^\alpha + N^d = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n=1}^{\infty} n\, d_{-n}^i d_n^i \tag{181}$$

♢ $\alpha' M^2 = N = 1, \;\; N^\alpha = 1, \;\; N^d = 0$

$$\alpha_{-1}^i \,|a\rangle \;\; ✓ \qquad\qquad \alpha_{-1}^i \,|\dot{a}\rangle \;\; ✗ \tag{182}$$

♢ $\alpha' M^2 = N = 1, \;\; N^\alpha = 0, \;\; N^d = 1$

$$d_{-1}^i \,|a\rangle \;\; ✗ \qquad\qquad d_{-1}^i \,|\dot{a}\rangle \;\; ✓ \tag{183}$$

Here $|a\rangle$ and $|\dot{a}\rangle$ are Majorana-Weyl spinors of opposite handless:

$$\Gamma_{11} \,|a\rangle = |a\rangle \,, \qquad \Gamma_{11} \,|\dot{a}\rangle = -\,|\dot{a}\rangle \tag{184}$$

Therefore, in the first massive level the states which survive the GSO projection are:

$$\begin{array}{cc} \alpha_{-1}^i \,|a\rangle & d_{-1}^i \,|\dot{a}\rangle \\ 8 \times 8 = 64 & 8 \times 8 = 64 \end{array} \tag{185}$$

These 128 fermionic states form an irreducible spinor representation of $Spin(9)$.

This massive supermultiplet in ten dimensions, consisting of 128 bosons and 128 fermions, is identical to the massless supergravity multiplet in 11 dimensions.

The second massive level of open strings [Problem 4.7]

- **NS sector**

$$\alpha' M^2 = N - \frac{1}{2} \qquad\qquad N = N^\alpha + N^b \tag{186}$$

♢ $N = 2, \ \alpha' M^2 = 3/2$

$$N^\alpha = 2,\ N^b = 0: \quad \alpha^i_{-1}\alpha^j_{-1}|0\rangle, \qquad \alpha^i_{-2}|0\rangle \tag{187}$$

$$N^\alpha = 1,\ N^b = 1: \quad \alpha^i_{-1}b^j_{-1/2}b^k_{-1/2}|0\rangle \tag{188}$$

$$N^\alpha = 0,\ N^b = 2: \quad b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}b^l_{-1/2}|0\rangle, \qquad \alpha^i_{-2}b^i_{-1/2}b^j_{-3/2}|0\rangle \tag{189}$$

The GSO projection will eliminate these states since they have minus G-parity.

♢ $N = \frac{5}{2}, \ \alpha' M^2 = 2$

$$N^\alpha = 2,\ N^b = \tfrac{1}{2}: \quad \alpha^i_{-1}\alpha^j_{-1}b^k_{-1/2}|0\rangle, \quad \alpha^i_{-2}b^j_{-1/2}|0\rangle \tag{190}$$

$$N^\alpha = 1,\ N^b = \tfrac{3}{2}: \quad \alpha^i_{-1}b^j_{-1/2}b^k_{-1/2}b^l_{-1/2}|0\rangle, \quad \alpha^i_{-1}b^j_{-3/2}|0\rangle \tag{191}$$

$$N^\alpha = 0,\ N^b = \tfrac{5}{2}: \quad b^{i_1}_{-1/2}b^{i_2}_{-1/2}b^{i_3}_{-1/2}b^{i_4}_{-1/2}b^{i_5}_{-1/2}|0\rangle, \quad b^i_{-1/2}b^j_{-1/2}b^k_{-3/2}|0\rangle, \quad b^k_{-5/2}|0\rangle \tag{192}$$

Total number of states is $288 + 64 + 448 + 64 + 56 + 244 = 1152$.

- **R sector**

$$\alpha' M^2 = N = N^\alpha + N^d \tag{193}$$

The second massive level: $\alpha' M^2 = N = 2$

$$N^\alpha = 2,\ N^d = 0: \quad \alpha^i_{-1}\alpha^j_{-1}\,|a\rangle\,, \quad \alpha^i_{-2}\,|a\rangle \quad ✓ \tag{194}$$

$$\alpha^i_{-1}\alpha^j_{-1}\,|\dot{a}\rangle\,, \quad \alpha^i_{-2}\,|\dot{a}\rangle \quad ✗ \tag{195}$$

$$N^\alpha = 1,\ N^d = 1: \quad \alpha^i_{-1}d^j_{-1}\,|a\rangle \quad ✗ \tag{196}$$

$$\alpha^i_{-1}d^j_{-1}\,|\dot{a}\rangle \quad ✓ \tag{197}$$

$$N^\alpha = 0,\ N^d = 2: \quad d^i_{-1}d^j_{-1}\,|a\rangle\,, \quad d^i_{-2}\,|a\rangle \quad ✓ \tag{198}$$

$$d^i_{-1}d^j_{-1}\,|\dot{a}\rangle\,, \quad d^i_{-2}\,|\dot{a}\rangle \quad ✗ \tag{199}$$

Total number of states is $(36 + 8 + 64 + 28 + 8) \times 8 = 1152$.

Therefore the numbers of bosonic and fermionic space-time states are equal to each other in this massive level.

The massless closed-string spectrum

In the NS sector, a bosonic physical state must satisfy the GSO condition

$$G\,|\phi\rangle = |\phi\rangle \qquad G = (-1)^{\sum_{r=1/2}^{\infty} b_{-r}\cdot b_r + 1} \tag{200}$$

The tachyon is eliminated.

In the R sector, a physical state must satisfy

$$\bar{\Gamma}\,|\psi\rangle = |\psi\rangle \qquad \bar{\Gamma} = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d_{-n}\cdot d_n} \tag{201}$$

Obviously, G-parity depends on the chirality of the ground state on which the states are built. Thus two different theories can be obtained depending on whether the G-parity of the left- and right-moving R sectors is the same or opposite.

- **Type IIB**

The left- and right-moving R-sector ground states have the same chirality, chosen to be positive for definiteness.

- **Type IIA**

The left- and right-moving R-sector ground states are chosen to have the opposite chirality.

- **Type IIB**

The left- and right-moving R-sector ground states have the same chirality. The massless states in the type IIB closed-string spectrum

$$
\begin{array}{rl}
|a\rangle_{\mathrm{R}} \otimes |a\rangle_{\mathrm{R}} & (1) \oplus (28) \oplus (35)_s \\
\tilde{b}^i_{-1/2}\,|0\rangle_{\mathrm{NS}} \otimes b^i_{-1/2}\,|0\rangle_{\mathrm{NS}} & (1) \oplus (28) \oplus (35)_v \\
\tilde{b}^i_{-1/2}\,|0\rangle_{\mathrm{NS}} \otimes |a\rangle_{\mathrm{R}} & (8)_c \oplus (56)_c \\
|a\rangle_{\mathrm{R}} \otimes b^i_{-1/2}\,|0\rangle_{\mathrm{NS}} & (8)_c \oplus (56)_c
\end{array}
\tag{202}
$$

where $|a\rangle_{\mathrm{R}}$ represents an eight-component spinor.

- **Type IIA**

The left- and right-moving R-sector ground states are chosen to have the opposite chirality. The massless states in the spectrum are given by

$$
\begin{array}{rl}
|\dot{a}\rangle_{\mathrm{R}} \otimes |a\rangle_{\mathrm{R}} & (8)_v \oplus (56)_v \\
\tilde{b}^i_{-1/2}\,|0\rangle_{\mathrm{NS}} \otimes b^i_{-1/2}\,|0\rangle_{\mathrm{NS}} & (1) \oplus (28) \oplus (35)_v \\
\tilde{b}^i_{-1/2}\,|0\rangle_{\mathrm{NS}} \otimes |a\rangle_{\mathrm{R}} & (8)_c \oplus (56)_c \\
|\dot{a}\rangle_{\mathrm{R}} \otimes b^i_{-1/2}\,|0\rangle_{\mathrm{NS}} & (8)_s \oplus (56)_s
\end{array}
\tag{203}
$$

Each of the four sectors contains $8 \times 8 = 64$ physical states.

The massless spectrum of type II closed-strings

- **NS-NS sector**

$$\tilde{b}^i_{-1/2}|0\rangle_{\rm NS}\otimes b^i_{-1/2}|0\rangle_{\rm NS} \tag{204}$$

This sector is the same for the type IIA and type IIB cases. These states form a $\mathbf{8}\otimes\mathbf{8}$ representation of $\mathrm{SO}(8)$. These decompose into three irreducible representations:

$$\begin{array}{ccccc} \texttt{traceless symmetric} & \oplus & \texttt{anti-symmetric} & \oplus & \texttt{singlet}\ (=\texttt{trace}) \\ G_{\mu\nu}(X) & & B_{\mu\nu}(X) & & \Phi(X) \\ \frac{1}{2}\times 9\times 8-1=35 & & \frac{1}{2}\times 8\times 7=28 & & 1 \end{array} \tag{205}$$

The spectrum contains a scalar called the dilaton (one state), an antisymmetric two-form gauge field (28 states) and a symmetric traceless rank-two tensor, the graviton (35 states).

- **NS-R and R-NS sectors**

$$\textbf{IIB:}\quad \begin{array}{c} \tilde{b}^i_{-1/2}|0\rangle_{\rm NS}\otimes|a\rangle_{\rm R} \\ |a\rangle_{\rm R}\otimes b^i_{-1/2}|0\rangle_{\rm NS} \end{array} \qquad \textbf{IIA:}\quad \begin{array}{c} \tilde{b}^i_{-1/2}|0\rangle_{\rm NS}\otimes|a\rangle_{\rm R} \\ |\dot{a}\rangle_{\rm R}\otimes b^i_{-1/2}|0\rangle_{\rm NS} \end{array} \tag{206}$$

Each of these sectors contains a spin 3/2 **gravitino** (56 states) and a spin 1/2 fermion called the **dilatino** (eight states). In the IIB case the two gravitinos have the same chirality, whereas in the type IIA case they have opposite chirality.

- **R-R sector**

$$\textbf{IIB:} \quad |a\rangle_{\mathrm{R}} \otimes |a\rangle_{\mathrm{R}} \qquad\qquad \textbf{IIA:} \quad |\dot{a}\rangle_{\mathrm{R}} \otimes |a\rangle_{\mathrm{R}} \tag{207}$$

These states are bosons obtained by tensoring a pair of Majorana-Weyl spinors. In the IIA case, the two Majorana-Weyl spinors have opposite chirality, and one obtains a one-form (vector) gauge field (eight states) and a three-form gauge field (56 states). In the IIB case the two Majorana-Weyl spinors have the same chirality, and one obtains a zero-form (that is, scalar) gauge field (one state), a two-form gauge field (28 states) and a four-form gauge field with a self-dual field strength (35 states).

Density of states for the bosonic string

The total number of open string states with $\alpha' M^2 = n - 1$, denoted by d_n, is conveniently defined as the coefficient of w^n in the generating function

$$Z(w) = \mathrm{tr}_{\mathcal{H}}\, w^N = \sum_{n=1}^{\infty} d_n w^n \quad \Longrightarrow \quad d_n = \frac{1}{2\pi} \oint \frac{G(w)}{w^{n+1}} dw \tag{208}$$

$w \neq 1$ being a complex number.

$$Z(w) = \mathrm{tr}_{\mathcal{H}}\, w^N = \prod_{i=1}^{D-2} \prod_{m=1}^{\infty} \mathrm{tr}\, w^{\alpha_{-m}^i \alpha_m^i} = \prod_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} w^{mn} \right)^{24} \tag{209}$$

Here we have used

$$\mathrm{tr}\, w^{\alpha_{-m}^i \alpha_m^i} = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \langle n, k |\, w^{\alpha_{-m}^i \alpha_m^i} \,| n, k \rangle = \sum_{n=0}^{\infty} w^{mn} \qquad | n, k \rangle \equiv (\alpha_{-k})^n \,|0\rangle \tag{210}$$

Therefore

$$Z(w) = \prod_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} w^{mn} \right)^{24} = \prod_{m=1}^{\infty} \left(\frac{1}{1 - w^m} \right)^{24} = \left[f(w) \right]^{-24} \tag{211}$$

where we already introduced the bosonic critical dimension and

$$f(w) = \prod_{m=1}^{\infty} \left(1 - w^m\right) \tag{212}$$

which is called the classical partition function.

Density of states for the RNS superstring

We would like to make a similar computation for the RNS superstring. The only difference is that we have to take account of the GSO projection in both sectors.

Using similar technique, we can obtain

$$Z_{\mathrm{NS}}(w) = \frac{1}{2\sqrt{w}}\left[\prod_{n=1}^{\infty}\left(\frac{1+w^{n-1/2}}{1-w^n}\right)^8 - \prod_{n=1}^{\infty}\left(\frac{1-w^{n-1/2}}{1-w^n}\right)^8\right] \tag{213}$$

for the NS sector and

$$Z_{\mathrm{R}}(w) = 8\prod_{n=1}^{\infty}\left(\frac{1+w^n}{1-w^n}\right)^8 \tag{214}$$

for the R sector.

In 1829, Carl Jacobi proved that

$$Z_{\mathrm{NS}}(w) = Z_{\mathrm{R}}(w). \tag{215}$$

Jacobi who referred to the identity as "a rather obscure formula" (aequatio identica satis abstrusa). It was named after this the abstruse identity. It is not hard to verify that

$$Z_{\mathrm{NS}}(w) = 8\big(1 + 16w + 144w^2 + \cdots\big) = Z_{\mathrm{R}}(w) \tag{216}$$

This observation does not constitute a proof of supersymmetry but, still, it is encouraging.

A brief introduction to SCFT and BRST

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