An Introduction to the Chern-Simons Theory

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Based on a talk presented at a Group Seminar at IHEP, CAS

RUC, Beijing May 20, 2014

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Backgrounds

- 1974, Shiing-Shen Chern and James Harris Simons

  Characteristic forms and geometric invariants, Annals Math., 99, 48 (1974)

- 1988, Edward Witten, Topological Quantum, Field Theory

- 1989, Edward Witten, Quantum field theory and the Jones polynomial

- 1989, Zhang-Hansson-Kivelson, Fractional Quantum Hall Effect

- Hihg T_c Superconductivity

- Topological strings

- Gravity

- M2-brane

- 2008, Ofer Aharony, Oren Bergman, Daniel Louis Jafferis and Juan Maldacena

  $\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals

- $\cdots$

Construction of Chern-Simons forms

Constructing a Chern-Simons form needs two ingredients: a symmetry group G in a certain representation and an odd-dimensional differentiable manifold M. The fundamental object in a gauge theory is the gauge connection A, a generalization of the vector potential. The connection A is a Lie algebra valued field that is also a one-form,

$$
\begin{aligned}
A &\equiv A_\mu dx^\mu &(1)\\
&= A_\mu^a T^a dx^\mu &(2)
\end{aligned}
$$

Here T^a, $a = 1, \ldots, N$ are generators of Lie algebra of the gauge group G and they satisfy

$$[T^a, T^b] = f^{abc} T^c \tag{3}$$

In this slide we take the following normalization

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab} \tag{4}$$

An element of the gauge group G acts on the connection A as

$$U\big(\alpha(x)\big) = e^{i\alpha^a(x)T^a} \in G \qquad \begin{aligned} A &\longrightarrow A^U = U^{-1}AU + U^{-1}dU \\ A_\mu &\longrightarrow A_\mu^U = U^{-1}A_\mu U + U^{-1}\partial_\mu U \end{aligned} \tag{5}$$

The connection is gauge-dependent and therefore not directly measurable. However, the curvature $F = dA + A \wedge A$ (gauge field strength in physics), transforms homogeneously,

$$\begin{aligned} F \;\longrightarrow\; F^U &= dA^U + A^U \wedge A^U \\ &= d\big(U^{-1}AU + U^{-1}dU\big) + \big(U^{-1}AU + U^{-1}dU\big) \wedge \big(U^{-1}AU + U^{-1}dU\big) \\ &= U^{-1}\big(dA + A \wedge A\big)U \;=\; U^{-1}\,F\,U \end{aligned} \tag{6}$$

Form the gauge transformation of the curvature, we find a very interesting $2n$-form

$$\mathrm{Tr}\left(F^n\right) \;\equiv\; \mathrm{Tr}\left(F \wedge \cdots \wedge F\right) \tag{7}$$

which is invariant under gauge transformation (5) or (6):

$$\mathrm{Tr}\left(F^n\right) \;\longrightarrow\; \mathrm{Tr}\left(F^n\right)^U \;\equiv\; \mathrm{Tr}\left(F^U \wedge \cdots \wedge F^U\right) \tag{8}$$

$$= \mathrm{Tr}\left(U^{-1}F \wedge \cdots \wedge FU\right) \tag{9}$$

$$= \mathrm{Tr}\left(F \wedge \cdots \wedge F\right) \;=\; \mathrm{Tr}\left(F^n\right) \tag{10}$$

In mathematics, invariants of this kind (or more generally, the trace of any polynomial in F), like the Euler or the Pontryagin forms, are called characteristic classes. In topology, a geometric or topological being can be easily constructed locally, but when they are generalized to the global, topological obstructions will be encountered. These topological obstructions are usually represented as a cohomology class on the manifold – characteristic class.

Chern form

We denote

$$P_{2n}(F) = \mathrm{Tr}\left(F^n\right) \tag{11}$$

this is a $2n$-form and also called nth Chern form. These Chern forms are invariant under the gauge transformation and they are all closed [Chern-Weil theorem]:

$$dP_{2n}(F) = \mathrm{Tr}\left(d_D F \wedge \cdots \wedge F + \cdots + F \wedge \cdots \wedge d_D F\right) = 0 \tag{12}$$

Here we have used (*Exercise 118* in [1])

$$\mathrm{Tr}\left(d_D F\right) = \mathrm{Tr}\left(dF + [A, F]\right) = d\,\mathrm{Tr}\left(F\right) + \mathrm{Tr}\left(A \wedge F - (-1)^{1\times 2} F \wedge A\right) = d\,\mathrm{Tr}\left(F\right) \tag{13}$$

and the Bianchi identity

$$\begin{aligned} d_D F &= dF + [A, F] \qquad (14) \\ &= d(dA + A \wedge A) + [A, dA + A \wedge A] \\ &= dA \wedge A - A \wedge dA + A \wedge dA - dA \wedge A + A \wedge \left(A \wedge A\right) - \left(A \wedge A\right) \wedge A \\ &= 0 \qquad (15) \end{aligned}$$

Chern class

We have proved that Chern forms are all closed. This means that the nth Chern form defines a cohomology class in $H^{2n}(M)$.

Under an infinitesimal variation δA of the connection A,

$$\begin{aligned}
\delta P_{2n}(F) &= \mathrm{Tr}\left(\delta F \wedge \cdots \wedge F + \cdots + F \wedge \cdots \wedge \delta F\right) \\
&= n\ \mathrm{Tr}\left(\delta F \wedge \cdots \wedge F\right) \\
&= n\ \mathrm{Tr}\left(d_D \delta A \wedge \cdots \wedge F\right) \\
&= n\ \mathrm{Tr}\left(d_D(\delta A \wedge \cdots \wedge F)\right) \\
&= n\, d\ \mathrm{Tr}\left(\delta A \wedge \cdots \wedge F\right)
\end{aligned} \tag{16}$$

the Chern form changes by an exact form. Here we have used the graded cyclic property, Bianchi identity (14), Eq. (12) and

$$\begin{aligned}
\delta F &= d(A + \delta A) + (A + \delta A) \wedge (A + \delta A) - \left(dA + A \wedge A\right) \\
&= d\delta A + A \wedge \delta A + \delta A \wedge A \\
&= d\delta A + [A,\, \delta A] \\
&= d_D \delta A
\end{aligned} \tag{17}$$

Chern class

Consider the following variation δA of the connection A,

$$\delta A = A' - A, \quad A_t = A + t\,\delta A \qquad \Longrightarrow \qquad A_{t=0} = A, \quad A_{t=1} = A' \tag{18}$$

the difference of Chern forms is exact [Chern-Weil theorem]:

$$\begin{aligned} P_{2n}(F') - P_{2n}(F) &= \mathrm{Tr}\left(F'^n\right) - \mathrm{Tr}\left(F^n\right) \\ &= \int_0^1 \frac{d}{dt}\,\mathrm{Tr}\left(F_t^n\right) dt \\ &= n \int_0^1 d\,\mathrm{Tr}\left(\delta A \wedge F_t^{n-1}\right) dt \\ &= d\left(n \int_0^1 \mathrm{Tr}\left(\delta A \wedge F_t^{n-1}\right) dt\right) \end{aligned} \tag{19}$$

We thus can define the nth Chern class $c_n(E)$ of the vector bundle E over M to be the cohomology class of $P_{2n}(F)$, where F is the curvature of any connection on E. These invariants are very important tool for classifying vector bundles, and show up throughout mathematics and physics.

When take proper normalization, their integral over a $2n$-dimensional compact orientable manifold M are integers

$$\int_M P_{2n} \sim \int_M c_n \in \mathbb{Z} \tag{20}$$

and they are all topological invariants. For example, $SU(2)$ gauge theory

$$k = -\frac{1}{8\pi^2} \int_{S^3} \mathrm{Tr}\left(F \wedge F\right) \in \mathbb{Z} \tag{21}$$

this is called the Pontryagin index (or winding number). More general, if E is a complex vector bundle,

$$C_n = \int c_n = \frac{\left(i/2\pi\right)^n}{n!} \int_M \mathrm{Tr}\left(F^n\right) \in \mathbb{Z} \tag{22}$$

is an integer, this also is called nth Chern number of the vector bundle E.

Chern-Simons forms

We have proved that Chern forms $P_{2n}(F)$ are all closed. By Poincare lemma, the nth Chern form $P_{2n}(F)$ can be locally expressed as the derivative of a $(2n-1)$-form,

$$P_{2n}(F) = dQ_{2n-1}(A, F) \tag{23}$$

It's important to note that this cannot be true globally. If $P_{2n} = dQ_{2n-1}$ globally on a manifold M without boundary, we would have

$$\int_M P_{2m} = \int_M dQ_{2m-1} = \int_{\partial M} Q_{2m-1} = 0 \tag{24}$$

where $\dim M = 2m$.

The $(2n-1)$-form $Q_{2n-1}(A, F)$ is called the Chern-Simons form of $P_{2n}(F)$. We can explicitly work out the Chern-Simons form. For example,

$$P_2(F) = \mathrm{Tr}\left(F\right) = \mathrm{Tr}\left(dA + A \wedge A\right) = d\,\mathrm{Tr}\left(A\right) \tag{25}$$

thus we get $Q_1(A, F) = \mathrm{Tr}(A)$.

Chern-Simons forms

Let $A_t = t\,A$ and let

$$F_t \;=\; dt \wedge A + t\,dA + t^2 A \wedge A \tag{26}$$

be the curvature of connection A_t. Then we have

$$\begin{aligned}
\mathrm{Tr}\left(F \wedge F\right) \;&=\; \int_0^1 \frac{d}{dt}\,\mathrm{Tr}\left(F_t \wedge F_t\right) dt \\
&=\; \int_0^1 \frac{d}{dt}\,\mathrm{Tr}\Big(\big(dt \wedge A + t\,dA + t^2 A \wedge A\big) \wedge \big(dt \wedge A + t\,dA + t^2 A \wedge A\big)\Big) dt \\
&=\; 2d \int_0^1 \mathrm{Tr}\Big(A \wedge \big(t\,dA + t^2 A \wedge A\big)\Big) dt \\
&=\; d\;\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \qquad (27)
\end{aligned}$$

Here the 3-form

$$\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \tag{28}$$

is famous Chern-Simons 3-form which is our concern.

It is nice that check directly that its exterior derivative is the second Chern form:

$$\begin{aligned} d\,\mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right) &= \mathrm{Tr}\left(dA\wedge dA+2dA\wedge A\wedge A\right) \\ &= \mathrm{Tr}\Big(\big(dA+A\wedge A\big)\wedge\big(dA+A\wedge A\big)\Big) \\ &= \mathrm{Tr}\left(F\wedge F\right) \end{aligned} \tag{29}$$

In five dimensions, the Chern-Simons 5-form is given by

$$Q_5(A,F) = \mathrm{Tr}\left(A\wedge dA\wedge dA+\frac{3}{2}A\wedge A\wedge A\wedge dA+\frac{3}{5}A\wedge A\wedge A\wedge A\wedge A\right) \tag{30}$$

$$= \mathrm{Tr}\left(F\wedge F\wedge A-\frac{1}{2}F\wedge A\wedge A\wedge A+\frac{1}{10}A\wedge A\wedge A\wedge A\wedge A\right) \tag{31}$$

Generally, in $(2n-1)$ dimensions the Chern-Simons form is given by

$$Q_{2n-1}(A,F) = \int_0^1 dt\,\mathrm{Tr}\left(A\wedge F_t^{n-1}\right), \tag{32}$$

$$F_t = t\,dA+t^2A\wedge A. \tag{33}$$

Non-Abelian Chern-Simons theories

The action of Chern-Simons theory is proportional to the integral of the Chern-Simons 3-form

$$S_{\mathrm{CS}} = \kappa \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \tag{34}$$

in $D = 3$ dimensions. Here we take the gauge group $G = U(N)$.

In components form, the above Non-Abelian Chern-Simons Lagrangian is

$$\mathcal{L}_{\mathrm{CS}} = \kappa \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right) \tag{35}$$

Under infinitesimal variations δA_μ of the gauge field, the change of the action is

$$\begin{aligned}
\delta S_{\mathrm{CS}} &= \kappa \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(\delta A_\mu \partial_\nu A_\rho + A_\mu \partial_\nu (\delta A_\rho) + \frac{2}{3}\delta A_\mu A_\nu A_\rho + \frac{2}{3} A_\mu \delta A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu \delta A_\rho\right) \\
&= \kappa \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(\delta A_\mu \partial_\nu A_\rho + A_\nu \partial_\rho (\delta A_\mu) + 2\delta A_\mu A_\nu A_\rho\right) \\
&= \kappa \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(\delta A_\mu \partial_\nu A_\rho - \delta A_\mu \partial_\rho A_\nu + 2\delta A_\mu A_\nu A_\rho\right) \\
&= \kappa \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(\delta A_\mu (\partial_\nu A_\rho - \partial_\rho A_\nu) + \delta A_\mu [A_\nu, A_\rho] + \partial_\rho (A_\nu \delta A_\mu)\right) \\
&= \kappa \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(\delta A_\mu F_{\nu\rho}\right).
\end{aligned} \tag{36}$$

where the gauge field strength is defined by

$$F = dA + A \wedge A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{37}$$

Alternative derivation

$$\begin{aligned}
\delta S &= S[A + \delta A] - S[A] &(38)\\
&= \kappa \int \mathrm{Tr}\left(A \wedge d\big(\delta A\big) + dA \wedge \delta A + 2\, A \wedge A \wedge \delta A + \mathcal{O}\big((\delta A)^2\big)\right)\\
&= \kappa \int \mathrm{Tr}\left(- d\big(A \wedge \delta A\big) + 2dA \wedge \delta A + 2\, A \wedge A \wedge \delta A + \mathcal{O}\big((\delta A)^2\big)\right)\\
&= 2\kappa \int \mathrm{Tr}\left(\big(dA + A \wedge A\big) \wedge \delta A\right)\\
&= 2\kappa \int \mathrm{Tr}\left(F \wedge \delta A\right) &(39)
\end{aligned}$$

This gives the equation of motion

$$F = dA + A \wedge A = 0 \tag{40}$$

The classical equations of motion are therefore satisfied if and only if **the curvature F vanishes everywhere, in which case the connection A is flat**. However, the quantum version of Chern-Simons theory is very interesting as we will see later.

Gauge transformation

The nonabelian gauge transformation U (which is an element of the gauge group) tarnsforms the gauge field as

$$A_\mu \longrightarrow A_\mu^U \equiv U^{-1}\big(A_\mu + \partial_\mu\big)U \tag{41}$$

First we consider an infinitesimal transformation, i.e.

$$U\big(\alpha(x)\big) \equiv e^{i\alpha^a t^a} = e^{i\alpha(x)} = 1 + i\alpha + \mathcal{O}(\alpha^2) \tag{42}$$

Thus the gauge transformation of the gauge field reduce to

$$A_\mu \longrightarrow A_\mu^U \equiv A_\mu - i[\alpha, A_\mu] + i\partial_\mu\alpha + \mathcal{O}(\alpha^2) \tag{43}$$

Under this infinitesimal transformation,

$$\begin{aligned}\mathcal{L}[A_\mu] \longrightarrow \mathcal{L}[A_\mu^U] &\equiv \kappa\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu^U\partial_\nu A_\rho^U + \frac{2}{3}A_\mu^U A_\nu^U A_\rho^U\right)\\ &= \kappa\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Bigg(\big(A_\mu - i[\alpha, A_\mu] + i\partial_\mu\alpha\big)\partial_\nu\big(A_\rho - i[\alpha, A_\rho] + i\partial_\rho\alpha\big)\\ &\quad + \frac{2}{3}\big(A_\mu - i[\alpha, A_\mu] + i\partial_\mu\alpha\big)\big(A_\nu - i[\alpha, A_\nu] + i\partial_\nu\alpha\big)\big(A_\rho - i[\alpha, A_\rho] + i\partial_\rho\alpha\big) + \mathcal{O}(\alpha^2)\Bigg)\end{aligned} \tag{44}$$

We calculate

$$
\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}[A_\mu^U] - \mathcal{L}[A_\mu] \\
&= \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(- iA_\mu\partial_\nu[\alpha, A_\rho] - i\big([\alpha, A_\mu] - \partial_\mu\alpha\big)\partial_\nu A_\rho - 2iA_\mu A_\nu\big([\alpha, A_\rho] - \partial_\rho\alpha\big) + \mathcal{O}(\alpha^2)\Big) \\
&= -i\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(A_\mu\partial_\nu[\alpha, A_\rho] + \big([\alpha, A_\mu] - \partial_\mu\alpha\big)\partial_\nu A_\rho - 2A_\mu A_\nu\partial_\rho\alpha + \mathcal{O}(\alpha^2)\Big) \\
&= -i\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(\partial_\nu\alpha\partial_\mu A_\rho\Big) + \mathcal{O}(\alpha^2) \\
&= -i\kappa\,\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\Big(\partial_\nu\alpha A_\rho\Big) + \mathcal{O}(\alpha^2)
\end{aligned}
\tag{45}
$$

In fact, we will see

$$
\begin{aligned}
\delta\mathcal{L} &= -i\kappa\,\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\Big(\partial_\nu\alpha A_\rho + \mathcal{O}(\alpha^2)\Big) \\
&= -\kappa\,\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\Big(\partial_\nu\big(1 + i\alpha + \mathcal{O}(\alpha^2)\big)\big(1 - i\alpha + \mathcal{O}(\alpha^2)\big)A_\rho\Big) \\
&= -\kappa\,\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\Big(\partial_\nu U U^{-1} A_\rho\Big)
\end{aligned}
\tag{46}
$$

Therefore, if we can neglect boundary terms then the corresponding Chern-Simons action is invariant under the infinitesimal gauge transformation.

Alternative derivation

Consider a smooth 1-parameter family of gauge transformations U_t,

$$t \in [0,1] \qquad \text{s.t.} \qquad U_{t=0} = 1, \quad U_{t=1} = U \tag{47}$$

Under such a gauge transformations U_t,

$$A \longrightarrow A_t = U_t^{-1} A U_t + U_t^{-1} dU_t \tag{48}$$

Then we claim

$$\frac{d}{dt} S(A_t) = 0. \tag{49}$$

With no loss of generality, it suffices to show this for $t = 0$:

$$\begin{aligned} \frac{dS_{\text{CS}}(A_t)}{dt}\Big|_{t=0} &= \kappa \frac{d}{dt} \int_M \text{Tr}\left(A_t \wedge dA_t + \frac{2}{3} A_t \wedge A_t \wedge A_t \right)\Big|_{t=0} \\ &= \kappa \int_M \text{Tr}\left(\frac{dA_t}{dt} \wedge dA_t + A_t \wedge \frac{d}{dt}(dA_t) + 2\frac{dA_t}{dt} \wedge A_t \wedge A_t \right)\Big|_{t=0} \end{aligned} \tag{50}$$

Notice that

$$0 = \frac{d}{dt}(U_t^{-1} U_t)_{t=0} = \left(\frac{dU_t^{-1}}{dt} U_t + U_t^{-1} \frac{dU_t}{dt} \right)_{t=0} = \left(\frac{dU_t^{-1}}{dt} + \frac{dU_t}{dt} \right)_{t=0} \tag{51}$$

So we can write

$$T = \frac{dU_t}{dt}\Big|_{t=0} \qquad \Longrightarrow \qquad \frac{dU_t^{-1}}{dt}\Big|_{t=0} = -T \tag{52}$$

Then we obtain

$$\begin{aligned}\frac{dA_t}{dt}\Big|_{t=0} &= \frac{d}{dt}\Big(U_t^{-1}AU_t + U_t^{-1}dU_t\Big)\Big|_{t=0} \\ &= \big(-TAU_t + U_t^{-1}AT - TdU_t + U_t^{-1}dT\big)\big|_{t=0} \\ &= \big(-TA + AT + dT\big) \end{aligned} \tag{53}$$

The first two terms of Eq. (50):

$$\begin{aligned} &\mathrm{Tr}\left(\frac{dA_t}{dt}\wedge dA_t + A_t \wedge \frac{d}{dt}(dA_t)\right)\Bigg|_{t=0} \\ =\, &\mathrm{Tr}\Big(\big(-TA + AT + dT\big)\wedge dA + A\wedge d\big(-TA+AT\big)\Big) \\ =\, &2\,\mathrm{Tr}\Big(2\big(-TA+AT\big)\wedge dA\Big) \;+\; d\;\mathrm{Tr}\Big(TdA + \big(-TA+AT\big)\wedge A\Big)\Big) \end{aligned} \tag{54}$$

Using above equations (53) and (54), then we have

$$\begin{aligned} \frac{dS_{\mathrm{CS}}(A_t)}{dt}\Big|_{t=0} &= 2\kappa\int_M \mathrm{Tr}\Big(\big(-TA+AT\big)\wedge dA + \big(-TA+AT+dT\big)\wedge A\wedge A\Big) \\ &= 2\kappa\int_M \mathrm{Tr}\Big(-TA\wedge dA + AT\wedge dA + dT\wedge A\wedge A\Big) \\ &= 2\kappa\int_M d\,\mathrm{Tr}\big(TA\wedge A\big) \\ &= 0 \end{aligned} \tag{55}$$

• **Large Gauge Transformation:**

$$\mathcal{L}[A_\mu] \;\to\; \mathcal{L}[A^U_\mu] \;\equiv\; \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A^U_\mu \partial_\nu A^U_\rho + \frac{2}{3}A^U_\mu A^U_\nu A^U_\rho\right) \tag{56}$$

Firstly we calculate the second term

$$\begin{aligned}
\mathcal{L}_2 \;\equiv\; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A^U_\mu A^U_\nu A^U_\rho\right) \\
= \; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\left(U^{-1}A_\mu U + U^{-1}\partial_\mu U\right)\left(U^{-1}A_\nu U + U^{-1}\partial_\nu U\right)\left(U^{-1}A_\rho U + U^{-1}\partial_\rho U\right)\right) \\
= \; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(\left(U^{-1}A_\mu U U^{-1}A_\nu U U^{-1}A_\rho U\right) + 3\,U^{-1}A_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U \\
& \qquad + 3\,U^{-1}\partial_\mu U U^{-1}A_\nu U U^{-1}A_\rho U + U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\Big) \\
= \; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(A_\mu A_\nu A_\rho + U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U \\
& \qquad + 3\,U^{-1}A_\mu \partial_\nu U U^{-1}\partial_\rho U + 3\,\partial_\mu U U^{-1}A_\nu A_\rho\Big) \\
= \; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu A_\nu A_\rho + U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right) \\
& + 2\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(U^{-1}A_\mu \partial_\nu U U^{-1}\partial_\rho U + \partial_\mu U U^{-1}A_\nu A_\rho\right)
\end{aligned} \tag{57}$$

The first term:

$$
\begin{aligned}
\mathcal{L}_1 &\equiv \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu^U \partial_\nu A_\rho^U\right) \\
&= \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\left(U^{-1}A_\mu U + U^{-1}\partial_\mu U\right)\partial_\nu\left(U^{-1}A_\rho U + U^{-1}\partial_\rho U\right)\right) \\
&= \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\left(U^{-1}A_\mu U + U^{-1}\partial_\mu U\right)\left(\partial_\nu U^{-1}A_\rho U + U^{-1}\partial_\nu A_\rho U + U^{-1}A_\rho\partial_\nu U + \partial_\nu U^{-1}\partial_\rho U\right)\right) \\
&= \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\left(U^{-1}A_\mu U U^{-1}\partial_\nu A_\rho U\right) + U^{-1}\partial_\mu U\partial_\nu U^{-1}\partial_\rho U\right. \\
&\qquad + U^{-1}A_\mu U\left(\partial_\nu U^{-1}A_\rho U + U^{-1}A_\rho\partial_\nu U + \partial_\nu U^{-1}\partial_\rho U\right)\Big) \\
&\qquad \left. + U^{-1}\partial_\mu U\left(\partial_\nu U^{-1}A_\rho U + U^{-1}\partial_\nu A_\rho U + U^{-1}A_\rho\partial_\nu U\right)\right) \\
&= \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right. \\
&\qquad + U^{-1}A_\mu U\partial_\nu U^{-1}A_\rho U + U^{-1}A_\mu U U^{-1}A_\rho\partial_\nu U + U^{-1}A_\mu U\partial_\nu U^{-1}\partial_\rho U\Big) \\
&\qquad \left. + U^{-1}\partial_\mu U\partial_\nu U^{-1}A_\rho U + U^{-1}\partial_\mu U U^{-1}\partial_\nu A_\rho U + U^{-1}\partial_\mu U U^{-1}A_\rho\partial_\nu U\right) \\
&= \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right. \\
&\qquad + A_\mu A_\rho\partial_\nu U U^{-1} + A_\mu A_\rho\partial_\nu U U^{-1} - A_\mu\partial_\nu U U^{-1}\partial_\rho U U^{-1} \\
&\qquad \left. - A_\rho\partial_\mu U U^{-1}\partial_\nu U U^{-1} + \partial_\mu U U^{-1}\partial_\nu A_\rho + A_\rho\partial_\nu U U^{-1}\partial_\mu U U^{-1}\right) \\
&= \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right. \\
&\qquad \left. - 2A_\mu A_\nu\partial_\rho U U^{-1} - 3A_\mu\partial_\nu U U^{-1}\partial_\rho U U^{-1} + \partial_\mu U U^{-1}\partial_\nu A_\rho\right)
\end{aligned} \tag{58}
$$

Thus we have

$$\begin{aligned}
\Delta\mathcal{L} &\equiv \mathcal{L}[A_\mu^U] - \mathcal{L}[A_\mu] \\
&= \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(-\frac{1}{3}U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U \\
&\qquad - 2A_\mu A_\nu \partial_\rho U U^{-1} - 3A_\mu \partial_\nu U U^{-1}\partial_\rho U U^{-1} + \partial_\mu U U^{-1}\partial_\nu A_\rho \\
&\qquad + 2\big(U^{-1}A_\mu \partial_\nu U U^{-1}\partial_\rho U + \partial_\mu U U^{-1} A_\nu A_\rho\big)\Big) \\
&= \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(-\frac{1}{3}U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U \\
&\qquad - A_\mu \partial_\nu U U^{-1}\partial_\rho U U^{-1} + \partial_\mu U U^{-1}\partial_\nu A_\rho\Big) \\
&= \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(-\frac{1}{3}U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U - \partial_\mu\Big(\partial_\nu U U^{-1} A_\rho\Big)\Big) \qquad (59)
\end{aligned}$$

The winding number of the group element $U \in G$ is

$$w = \frac{1}{24\pi^2}\int d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\Big) \in \mathbb{Z}, \qquad \pi_3(U(N)) \cong \mathbb{Z} \qquad (60)$$

Final we obtain the change of the action under the large gauge transformation

$$S[A] \longrightarrow S[A^U] = S[A] - \kappa\,8\pi^2\,w \qquad (61)$$

In path integral quantization formalism, one needs the measure

$$[\mathcal{D}A]\, e^{iS[A]} \tag{62}$$

invariant under the gauge transformation. Firstly notice that the Lebesgue measure $\mathcal{D}A$ is gauge invariant:

$$[\mathcal{D}A] \equiv \prod_{a,\mu,x} dA_\mu^a(x) \qquad \Longrightarrow \qquad [\mathcal{D}A^U] = [\mathcal{D}A] \tag{63}$$

So we must require

$$S[A] \longrightarrow S[A^U] = S[A] - \kappa\, 8\pi^2\, w = S[A] + 2\pi N, \qquad N \in \mathbb{N} \tag{64}$$

This gives

$$\kappa = \frac{k}{4\pi}, \qquad S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad k \in \mathbb{N}^+ \tag{65}$$

Here the positive integer k is called the Chern-Simons level.

As pointed out by Witten, consistency of quantum field theory does not quite require the single-valuedness of S, but only of $\exp(iS)$ [10].

Topological invariants

Since the action of Chern-Simons theory

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{66}$$

does not involve the metric, the resulting quantum theory is topological, at least formally. That is, the theory contains topological invariants. In particular, the partition function

$$Z(M) = \int [\mathcal{D}A]\, e^{iS[A]} \tag{67}$$

should define a topological invariant of the manifold M without boundary. A detailed analysis shows that this is in fact the case, with an extra subtlety: the invariant depends not only on the three-manifold but also on a choice of framing [10, 11].

The observables of Chern-Simons theory are the n-point correlation functions of gauge-invariant operators. The most often studied class of gauge invariant operators are Wilson loops. A Wilson loop is the holonomy around a loop in M, traced in a given representation R of G. More concretely, given an irreducible representation R and a loop γ in M, one may define the Wilson loop $W_R(\gamma)$ by

$$W_R(\gamma) \equiv \mathrm{Tr}_R\, \mathcal{P}\, \exp\left(i\oint_\gamma A\right) \tag{68}$$

where $\mathcal{P}$ is the path-ordered.

Summary

The action of Chern-Simons theory in $D = 3$ dimensions is given by

$$S_{\rm CS} = \frac{k}{4\pi} \int {\rm Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{69}$$

- **Topological origin** The Chern-Simons form is related to a topological density in $2n$ dimensions known as a characteristic class Q_{2n-1}, through

$$P_{2n}(F) = dQ_{2n-1}(A, F) \tag{70}$$

$$ {\rm Tr} \left(F \wedge F \right) = d\,{\rm Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{71}$$

- **No metric required** As noticed by Witten [10], since this action does not involve the metric, the resulting quantum theory is topological, at least formally.
- **Orientation-preserving diffeomorphisms**
- **Gauge invariance** The Chern-Simons action is invariant under the infinitesimal gauge transformation. Under the large gauge transformation, the change of the action

$$S[A] \longrightarrow S[A^U] = S[A] - 2\pi\, k\, N, \qquad k \in \mathbb{N}^+ \tag{72}$$

The positive integer k is called the level of the quantum Chern-Simons theory.

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