# An Introduction to the Chern-Simons Theory

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# Construction of Chern-Simons forms

Constructing a Chern-Simons form needs two ingredients: a symmetry group $G$ in a certain representation and an odd-dimensional differentiable manifold $M$. The fundamental object in a gauge theory is the gauge connection $A$, a generalization of the vector potential. The connection $A$ is a Lie algebra valued field that is also a one-form,

$$\begin{aligned} A &\equiv A_\mu dx^\mu && (1) \\ &= A_\mu^a T^a dx^\mu && (2) \end{aligned}$$

Here $T^a$, $a = 1, \ldots, N$ are generators of Lie algebra of the gauge group $G$ and they satisfy

$$[T^a, T^b] = f^{abc} T^c \tag{3}$$

In this slide we take the following normalization

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab} \tag{4}$$

An element of the gauge group $G$ acts on the connection $A$ as

$$U\big(\alpha(x)\big) = e^{i\alpha^a(x)T^a} \in G \qquad \begin{aligned} A &\longrightarrow A^U = U^{-1}AU + U^{-1}dU \\ A_\mu &\longrightarrow A_\mu^U = U^{-1}A_\mu U + U^{-1}\partial_\mu U \end{aligned} \tag{5}$$

The connection is gauge-dependent and therefore not directly measurable. However, the curvature $F = dA + A \wedge A$ (gauge field strength in physics), transforms homogeneously,

$$\begin{aligned} F \;\longrightarrow\; F^U \;&=\; dA^U + A^U \wedge A^U \\ &=\; d\big(U^{-1}AU + U^{-1}dU\big) + \big(U^{-1}AU + U^{-1}dU\big) \wedge \big(U^{-1}AU + U^{-1}dU\big) \\ &=\; U^{-1}\big(dA + A \wedge A\big)U \;=\; U^{-1}\,F\,U \end{aligned} \tag{6}$$

Form the gauge transformation of the curvature, we find a very interesting $2n$-form

$$\mathrm{Tr}\left(F^n\right) \;\equiv\; \mathrm{Tr}\left(F \wedge \cdots \wedge F\right) \tag{7}$$

which is invariant under gauge transformation (5) or (6):

$$\mathrm{Tr}\left(F^n\right) \;\longrightarrow\; \mathrm{Tr}\left(F^n\right)^U \;\equiv\; \mathrm{Tr}\left(F^U \wedge \cdots \wedge F^U\right) \tag{8}$$

$$= \mathrm{Tr}\left(U^{-1}F \wedge \cdots \wedge FU\right) \tag{9}$$

$$= \mathrm{Tr}\left(F \wedge \cdots \wedge F\right) \;=\; \mathrm{Tr}\left(F^n\right) \tag{10}$$

In mathematics, invariants of this kind (or more generally, the trace of any polynomial in $F$), like the Euler or the Pontryagin forms, are called characteristic classes. In topology, a geometric or topological being can be easily constructed locally, but when they are generalized to the global, topological obstructions will be encountered. These topological obstructions are usually represented as a cohomology class on the manifold – characteristic class.

## Chern form

We denote

$$P_{2n}(F) = \mathrm{Tr}\left(F^n\right) \tag{11}$$

this is a $2n$-form and also called $n$th Chern form. These Chern forms are invariant under the gauge transformation and they are all closed [Chern-Weil theorem]:

$$dP_{2n}(F) = \mathrm{Tr}\left(d_D F \wedge \cdots \wedge F + \cdots + F \wedge \cdots \wedge d_D F\right) = 0 \tag{12}$$

Here we have used (*Exercise 118* in [1])

$$\mathrm{Tr}\left(d_D F\right) = \mathrm{Tr}\left(dF + [A, F]\right) = d\,\mathrm{Tr}\left(F\right) + \mathrm{Tr}\left(A \wedge F - (-1)^{1\times 2} F \wedge A\right) = d\,\mathrm{Tr}\left(F\right) \tag{13}$$

and the Bianchi identity

$$\begin{aligned} d_D F &= dF + [A, F] \qquad (14)\\ &= d(dA + A \wedge A) + [A, dA + A \wedge A] \\ &= dA \wedge A - A \wedge dA + A \wedge dA - dA \wedge A + A \wedge \left(A \wedge A\right) - \left(A \wedge A\right) \wedge A \\ &= 0 \qquad (15) \end{aligned}$$

## Chern class

We have proved that Chern forms are all closed. This means that the $n$th Chern form defines a cohomology class in $H^{2n}(M)$.

Under an infinitesimal variation $\delta A$ of the connection $A$,

$$\begin{aligned} \delta P_{2n}(F) &= \mathrm{Tr}\left(\delta F \wedge \cdots \wedge F + \cdots + F \wedge \cdots \wedge \delta F\right) \\ &= n\ \mathrm{Tr}\left(\delta F \wedge \cdots \wedge F\right) \\ &= n\ \mathrm{Tr}\left(d_D \delta A \wedge \cdots \wedge F\right) \\ &= n\ \mathrm{Tr}\left(d_D(\delta A \wedge \cdots \wedge F)\right) \\ &= n\, d\ \mathrm{Tr}\left(\delta A \wedge \cdots \wedge F\right) \end{aligned} \tag{16}$$

the Chern form changes by an exact form. Here we have used the graded cyclic property, Bianchi identity (14), Eq. (12) and

$$\begin{aligned} \delta F &= d(A+\delta A) + (A+\delta A)\wedge(A+\delta A) - \left(dA + A\wedge A\right) \\ &= d\delta A + A\wedge\delta A + \delta A \wedge A \\ &= d\delta A + [A,\, \delta A] \\ &= d_D \delta A \end{aligned} \tag{17}$$

## Chern class

Consider the following variation $\delta A$ of the connection $A$,

$$\delta A = A' - A, \quad A_t = A + t\,\delta A \qquad \Longrightarrow \qquad A_{t=0} = A, \quad A_{t=1} = A' \tag{18}$$

the difference of Chern forms is exact [Chern-Weil theorem]:

$$\begin{aligned} P_{2n}(F') - P_{2n}(F) &= \mathrm{Tr}\left(F'^n\right) - \mathrm{Tr}\left(F^n\right) \\ &= \int_0^1 \frac{d}{dt}\,\mathrm{Tr}\left(F_t^n\right) dt \\ &= n \int_0^1 d\,\mathrm{Tr}\left(\delta A \wedge F_t^{n-1}\right) dt \\ &= d\left(n \int_0^1 \mathrm{Tr}\left(\delta A \wedge F_t^{n-1}\right) dt\right) \end{aligned} \tag{19}$$

We thus can define the $n$th Chern class $c_n(E)$ of the vector bundle $E$ over $M$ to be the cohomology class of $P_{2n}(F)$, where $F$ is the curvature of any connection on $E$. These invariants are very important tool for classifying vector bundles, and show up throughout mathematics and physics.

When take proper normalization, their integral over a $2n$-dimensional compact orientable manifold $M$ are integers

$$\int_M P_{2n} \sim \int_M c_n \in \mathbb{Z} \tag{20}$$

and they are all topological invariants. For example, $SU(2)$ gauge theory

$$k = -\frac{1}{8\pi^2} \int_{S^3} \mathrm{Tr}\left(F \wedge F\right) \in \mathbb{Z} \tag{21}$$

this is called the Pontryagin index (or winding number). More general, if $E$ is a complex vector bundle,

$$C_n = \int c_n = \frac{\left(i/2\pi\right)^n}{n!} \int_M \mathrm{Tr}\left(F^n\right) \in \mathbb{Z} \tag{22}$$

is an integer, this also is called $n$th Chern number of the vector bundle $E$.

## Chern-Simons forms

We have proved that Chern forms $P_{2n}(F)$ are all closed. By Poincare lemma, the $n$th Chern form $P_{2n}(F)$ can be locally expressed as the derivative of a $(2n-1)$-form,

$$P_{2n}(F) = dQ_{2n-1}(A, F) \tag{23}$$

It's important to note that this cannot be true globally. If $P_{2n} = dQ_{2n-1}$ globally on a manifold $M$ without boundary, we would have

$$\int_M P_{2m} = \int_M dQ_{2m-1} = \int_{\partial M} Q_{2m-1} = 0 \tag{24}$$

where $\dim M = 2m$.

The $(2n-1)$-form $Q_{2n-1}(A, F)$ is called the Chern-Simons form of $P_{2n}(F)$. We can explicitly work out the Chern-Simons form. For example,

$$P_2(F) = \mathrm{Tr}\left(F\right) = \mathrm{Tr}\left(dA + A \wedge A\right) = d\ \mathrm{Tr}\left(A\right) \tag{25}$$

thus we get $Q_1(A, F) = \mathrm{Tr}(A)$.

## Chern-Simons forms

Let $A_t = t\,A$ and let

$$F_t \;=\; dt \wedge A + t\,dA + t^2 A \wedge A \tag{26}$$

be the curvature of connection $A_t$. Then we have

$$\begin{aligned}
\mathrm{Tr}\left(F \wedge F\right) \;&=\; \int_0^1 \frac{d}{dt}\,\mathrm{Tr}\left(F_t \wedge F_t\right) dt \\
&=\; \int_0^1 \frac{d}{dt}\,\mathrm{Tr}\Big(\left(dt \wedge A + t\,dA + t^2 A \wedge A\right) \wedge \left(dt \wedge A + t\,dA + t^2 A \wedge A\right)\Big) dt \\
&=\; 2d \int_0^1 \mathrm{Tr}\Big(A \wedge \left(t\,dA + t^2 A \wedge A\right)\Big) dt \\
&=\; d\;\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)
\end{aligned} \tag{27}$$

Here the 3-form

$$\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \tag{28}$$

is famous Chern-Simons 3-form which is our concern.

It is nice that check directly that its exterior derivative is the second Chern form:

$$\begin{aligned} d\,\mathrm{Tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big) &= \mathrm{Tr}\Big(dA \wedge dA + 2dA \wedge A \wedge A\Big) \\ &= \mathrm{Tr}\Big(\big(dA + A \wedge A\big) \wedge \big(dA + A \wedge A\big)\Big) \\ &= \mathrm{Tr}\big(F \wedge F\big) \end{aligned} \tag{29}$$

In five dimensions, the Chern-Simons 5-form is given by

$$Q_5(A, F) = \mathrm{Tr}\Big(A \wedge dA \wedge dA + \frac{3}{2} A \wedge A \wedge A \wedge dA + \frac{3}{5} A \wedge A \wedge A \wedge A \wedge A\Big) \tag{30}$$

$$= \mathrm{Tr}\Big(F \wedge F \wedge A - \frac{1}{2} F \wedge A \wedge A \wedge A + \frac{1}{10} A \wedge A \wedge A \wedge A \wedge A\Big) \tag{31}$$

Generally, in $(2n-1)$ dimensions the Chern-Simons form is given by

$$Q_{2n-1}(A, F) = \int_0^1 dt\,\mathrm{Tr}\big(A \wedge F_t^{n-1}\big), \tag{32}$$

$$F_t = t\,dA + t^2 A \wedge A. \tag{33}$$

## Non-Abelian Chern-Simons theories

The action of Chern-Simons theory is proportional to the integral of the Chern-Simons 3-form

$$S_{\rm CS} = \kappa \int {\rm Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{34}$$

in $D = 3$ dimensions. Here we take the gauge group $G = U(N)$.

In components form, the above Non-Abelian Chern-Simons Lagrangian is

$$\mathcal{L}_{\rm CS} = \kappa \epsilon^{\mu\nu\rho} \, {\rm Tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) \tag{35}$$

Under infinitesimal variations $\delta A_\mu$ of the gauge field, the change of the action is

$$\begin{aligned} \delta S_{\rm CS} &= \kappa \int d^3x \, \epsilon^{\mu\nu\rho} \, {\rm Tr} \left( \delta A_\mu \partial_\nu A_\rho + A_\mu \partial_\nu (\delta A_\rho) + \frac{2}{3} \delta A_\mu A_\nu A_\rho + \frac{2}{3} A_\mu \delta A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu \delta A_\rho \right) \\ &= \kappa \int d^3x \, \epsilon^{\mu\nu\rho} \, {\rm Tr} \left( \delta A_\mu \partial_\nu A_\rho + A_\nu \partial_\rho (\delta A_\mu) + 2 \delta A_\mu A_\nu A_\rho \right) \\ &= \kappa \int d^3x \, \epsilon^{\mu\nu\rho} \, {\rm Tr} \left( \delta A_\mu \partial_\nu A_\rho - \delta A_\mu \partial_\rho A_\nu + 2 \delta A_\mu A_\nu A_\rho \right) \\ &= \kappa \int d^3x \, \epsilon^{\mu\nu\rho} \, {\rm Tr} \left( \delta A_\mu \big( \partial_\nu A_\rho - \partial_\rho A_\nu \big) + \delta A_\mu [A_\nu, A_\rho] + \partial_\rho (A_\nu \delta A_\mu) \right) \\ &= \kappa \int d^3x \, \epsilon^{\mu\nu\rho} \, {\rm Tr} \left( \delta A_\mu F_{\nu\rho} \right). \end{aligned} \tag{36}$$

where the gauge field strength is defined by

$$F = dA + A \wedge A = \frac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}, \qquad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}] \tag{37}$$

## Alternative derivation

$$\begin{aligned} \delta S &= S[A + \delta A] - S[A] \qquad (38) \\ &= \kappa \int \mathrm{Tr} \Big( A \wedge d\big(\delta A\big) + dA \wedge \delta A + 2\, A \wedge A \wedge \delta A + \mathcal{O}\big((\delta A)^2\big) \Big) \\ &= \kappa \int \mathrm{Tr} \Big( - d\big(A \wedge \delta A\big) + 2 dA \wedge \delta A + 2\, A \wedge A \wedge \delta A + \mathcal{O}\big((\delta A)^2\big) \Big) \\ &= 2\kappa \int \mathrm{Tr} \Big( \big(dA + A \wedge A\big) \wedge \delta A \Big) \\ &= 2\kappa \int \mathrm{Tr} \left( F \wedge \delta A \right) \qquad (39) \end{aligned}$$

This gives the equation of motion

$$F = dA + A \wedge A = 0 \tag{40}$$

The classical equations of motion are therefore satisfied if and only if **the curvature $F$ vanishes everywhere, in which case the connection $A$ is flat**. However, the quantum version of Chern-Simons theory is very interesting as we will see later.

## Gauge transformation

The nonabelian gauge transformation $U$ (which is an element of the gauge group) tarnsforms the gauge field as

$$A_\mu \longrightarrow A_\mu^U \equiv U^{-1}\big(A_\mu + \partial_\mu\big)U \tag{41}$$

First we consider an infinitesimal transformation, i.e.

$$U\big(\alpha(x)\big) \equiv e^{i\alpha^a t^a} = e^{i\alpha(x)} = 1 + i\alpha + \mathcal{O}(\alpha^2) \tag{42}$$

Thus the gauge transformation of the gauge field reduce to

$$A_\mu \longrightarrow A_\mu^U \equiv A_\mu - i[\alpha, A_\mu] + i\partial_\mu\alpha + \mathcal{O}(\alpha^2) \tag{43}$$

Under this infinitesimal transformation,

$$\begin{aligned}\mathcal{L}[A_\mu] \longrightarrow \mathcal{L}[A_\mu^U] &\equiv \kappa\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu^U\partial_\nu A_\rho^U + \frac{2}{3}A_\mu^U A_\nu^U A_\rho^U\right)\\ &= \kappa\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Bigg(\big(A_\mu - i[\alpha, A_\mu] + i\partial_\mu\alpha\big)\partial_\nu\big(A_\rho - i[\alpha, A_\rho] + i\partial_\rho\alpha\big)\\ &\quad + \frac{2}{3}\big(A_\mu - i[\alpha, A_\mu] + i\partial_\mu\alpha\big)\big(A_\nu - i[\alpha, A_\nu] + i\partial_\nu\alpha\big)\big(A_\rho - i[\alpha, A_\rho] + i\partial_\rho\alpha\big) + \mathcal{O}(\alpha^2)\Bigg)\end{aligned} \tag{44}$$

We calculate

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}[A_\mu^U] - \mathcal{L}[A_\mu] \\
&= \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big( -iA_\mu\partial_\nu[\alpha, A_\rho] - i\big([\alpha, A_\mu] - \partial_\mu\alpha\big)\partial_\nu A_\rho - 2iA_\mu A_\nu\big([\alpha, A_\rho] - \partial_\rho\alpha\big) + \mathcal{O}(\alpha^2)\Big) \\
&= -i\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(A_\mu\partial_\nu[\alpha, A_\rho] + \big([\alpha, A_\mu] - \partial_\mu\alpha\big)\partial_\nu A_\rho - 2A_\mu A_\nu\partial_\rho\alpha + \mathcal{O}(\alpha^2)\Big) \\
&= -i\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(\partial_\nu\alpha\partial_\mu A_\rho\Big) + \mathcal{O}(\alpha^2) \\
&= -i\kappa\,\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\Big(\partial_\nu\alpha A_\rho\Big) + \mathcal{O}(\alpha^2) \qquad (45)
\end{aligned}$$

In fact, we will see

$$\begin{aligned}
\delta\mathcal{L} &= -i\kappa\,\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\Big(\partial_\nu\alpha A_\rho + \mathcal{O}(\alpha^2)\Big) \\
&= -\kappa\,\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\Big(\partial_\nu\big(1 + i\alpha + \mathcal{O}(\alpha^2)\big)\big(1 - i\alpha + \mathcal{O}(\alpha^2)\big)A_\rho\Big) \\
&= -\kappa\,\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\Big(\partial_\nu U U^{-1} A_\rho\Big) \qquad (46)
\end{aligned}$$

Therefore, if we can neglect boundary terms then the corresponding Chern-Simons action is invariant under the infinitesimal gauge transformation.

## Alternative derivation

Consider a smooth 1-parameter family of gauge transformations $U_t$,

$$t \in [0,1] \qquad \text{s.t.} \qquad U_{t=0} = 1, \quad U_{t=1} = U \tag{47}$$

Under such a gauge transformations $U_t$,

$$A \longrightarrow A_t = U_t^{-1} A U_t + U_t^{-1} dU_t \tag{48}$$

Then we claim

$$\frac{d}{dt} S(A_t) = 0. \tag{49}$$

With no loss of generality, it suffices to show this for $t = 0$:

$$\begin{aligned} \frac{dS_{\text{CS}}(A_t)}{dt}\Big|_{t=0} &= \kappa \frac{d}{dt} \int_M \text{Tr}\left( A_t \wedge dA_t + \frac{2}{3} A_t \wedge A_t \wedge A_t \right)\Big|_{t=0} \\ &= \kappa \int_M \text{Tr}\left( \frac{dA_t}{dt} \wedge dA_t + A_t \wedge \frac{d}{dt}(dA_t) + 2\frac{dA_t}{dt} \wedge A_t \wedge A_t \right)\Big|_{t=0} \end{aligned} \tag{50}$$

Notice that

$$0 = \frac{d}{dt}(U_t^{-1} U_t)_{t=0} = \left( \frac{dU_t^{-1}}{dt} U_t + U_t^{-1} \frac{dU_t}{dt} \right)_{t=0} = \left( \frac{dU_t^{-1}}{dt} + \frac{dU_t}{dt} \right)_{t=0} \tag{51}$$

So we can write

$$T = \frac{dU_t}{dt}\Big|_{t=0} \qquad \Longrightarrow \qquad \frac{dU_t^{-1}}{dt}\Big|_{t=0} = -T \tag{52}$$

Then we obtain

$$\begin{aligned}
\left.\frac{dA_t}{dt}\right|_{t=0} &= \frac{d}{dt}\Big(U_t^{-1}AU_t + U_t^{-1}dU_t\Big)\Big|_{t=0} \\
&= \big(-TAU_t + U_t^{-1}AT - TdU_t + U_t^{-1}dT\big)\big|_{t=0} \\
&= \big(-TA + AT + dT\big) \qquad (53)
\end{aligned}$$

The first two terms of Eq. (50):

$$\begin{aligned}
&\operatorname{Tr}\left(\frac{dA_t}{dt} \wedge dA_t + A_t \wedge \frac{d}{dt}(dA_t)\right)\Bigg|_{t=0} \\
&= \operatorname{Tr}\Big(\big(-TA + AT + dT\big) \wedge dA + A \wedge d\big(-TA + AT\big)\Big) \\
&= 2\operatorname{Tr}\Big(2\big(-TA + AT\big) \wedge dA\Big) \ + \ d\operatorname{Tr}\Big(TdA + \big(-TA + AT\big) \wedge A\Big) \qquad (54)
\end{aligned}$$

Using above equations (53) and (54), then we have

$$\begin{aligned}
\left.\frac{dS_{\mathrm{CS}}(A_t)}{dt}\right|_{t=0} &= 2\kappa \int_M \operatorname{Tr}\Big(\big(-TA + AT\big) \wedge dA + \big(-TA + AT + dT\big) \wedge A \wedge A\Big) \\
&= 2\kappa \int_M \operatorname{Tr}\Big(-TA \wedge dA + AT \wedge dA + dT \wedge A \wedge A\Big) \\
&= 2\kappa \int_M d\operatorname{Tr}\big(TA \wedge A\big) \\
&= 0 \qquad (55)
\end{aligned}$$

- **Large Gauge Transformation:**

$$\mathcal{L}[A_\mu] \;\to\; \mathcal{L}[A_\mu^U] \;\equiv\; \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu^U\partial_\nu A_\rho^U + \frac{2}{3}A_\mu^U A_\nu^U A_\rho^U\right) \tag{56}$$

Firstly we calculate the second term

$$\begin{aligned}
\mathcal{L}_2 \;\equiv\; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu^U A_\nu^U A_\rho^U\right)\\
=\; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\left(U^{-1}A_\mu U + U^{-1}\partial_\mu U\right)\left(U^{-1}A_\nu U + U^{-1}\partial_\nu U\right)\left(U^{-1}A_\rho U + U^{-1}\partial_\rho U\right)\right)\\
=\; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\left(U^{-1}A_\mu U U^{-1}A_\nu U U^{-1}A_\rho U\right) + 3\,U^{-1}A_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right.\\
& \qquad\qquad \left. + 3\,U^{-1}\partial_\mu U U^{-1}A_\nu U U^{-1}A_\rho U + U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right)\\
=\; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu A_\nu A_\rho + U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right.\\
& \qquad\qquad \left. + 3\,U^{-1}A_\mu\partial_\nu U U^{-1}\partial_\rho U + 3\,\partial_\mu U U^{-1}A_\nu A_\rho\right)\\
=\; & \frac{2}{3}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu A_\nu A_\rho + U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right)\\
& + 2\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(U^{-1}A_\mu\partial_\nu U U^{-1}\partial_\rho U + \partial_\mu U U^{-1}A_\nu A_\rho\right)
\end{aligned} \tag{57}$$

The first term:

$$
\begin{aligned}
\mathcal{L}_1 \equiv\ & \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A^U_\mu \partial_\nu A^U_\rho\right) \\
=\ & \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\left(U^{-1}A_\mu U + U^{-1}\partial_\mu U\right)\partial_\nu\left(U^{-1}A_\rho U + U^{-1}\partial_\rho U\right)\right) \\
=\ & \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\left(U^{-1}A_\mu U + U^{-1}\partial_\mu U\right)\left(\partial_\nu U^{-1}A_\rho U + U^{-1}\partial_\nu A_\rho U + U^{-1}A_\rho\partial_\nu U + \partial_\nu U^{-1}\partial_\rho U\right)\right) \\
=\ & \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(\left(U^{-1}A_\mu U U^{-1}\partial_\nu A_\rho U\right) + U^{-1}\partial_\mu U\partial_\nu U^{-1}\partial_\rho U \\
& \qquad + U^{-1}A_\mu U\left(\partial_\nu U^{-1}A_\rho U + U^{-1}A_\rho\partial_\nu U + \partial_\nu U^{-1}\partial_\rho U\right)\Big) \\
& \qquad + U^{-1}\partial_\mu U\left(\partial_\nu U^{-1}A_\rho U + U^{-1}\partial_\nu A_\rho U + U^{-1}A_\rho\partial_\nu U\right)\Big) \\
=\ & \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(A_\mu\partial_\nu A_\rho - U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U \\
& \qquad + U^{-1}A_\mu U\partial_\nu U^{-1}A_\rho U + U^{-1}A_\mu U U^{-1}A_\rho\partial_\nu U + U^{-1}A_\mu U\partial_\nu U^{-1}\partial_\rho U\Big) \\
& \qquad + U^{-1}\partial_\mu U\partial_\nu U^{-1}A_\rho U + U^{-1}\partial_\mu U U^{-1}\partial_\nu A_\rho U + U^{-1}\partial_\mu U U^{-1}A_\rho\partial_\nu U\Big) \\
=\ & \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(A_\mu\partial_\nu A_\rho - U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U \\
& \qquad + A_\mu A_\rho\partial_\nu U U^{-1} + A_\mu A_\rho\partial_\nu U U^{-1} - A_\mu\partial_\nu U U^{-1}\partial_\rho U U^{-1} \\
& \qquad - A_\rho\partial_\mu U U^{-1}\partial_\nu U U^{-1} + \partial_\mu U U^{-1}\partial_\nu A_\rho + A_\rho\partial_\nu U U^{-1}\partial_\mu U U^{-1}\Big) \\
=\ & \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big({\color{blue}A_\mu\partial_\nu A_\rho}{\color{red}-U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U} \\
& \qquad - 2A_\mu A_\nu\partial_\rho U U^{-1} - 3A_\mu\partial_\nu U U^{-1}\partial_\rho U U^{-1} + \partial_\mu U U^{-1}\partial_\nu A_\rho\Big)
\end{aligned}
\tag{58}
$$

Thus we have

$$
\begin{aligned}
\Delta\mathcal{L} &\equiv \mathcal{L}[A_\mu^U] - \mathcal{L}[A_\mu] \\
&= \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\bigg(-\frac{1}{3}U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}\partial_\rho U \\
&\qquad - 2A_\mu A_\nu\partial_\rho UU^{-1} - 3A_\mu\partial_\nu UU^{-1}\partial_\rho UU^{-1} + \partial_\mu UU^{-1}\partial_\nu A_\rho \\
&\qquad + 2\big(U^{-1}A_\mu\partial_\nu UU^{-1}\partial_\rho U + \partial_\mu UU^{-1}A_\nu A_\rho\big)\bigg) \\
&= \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\bigg(-\frac{1}{3}U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}\partial_\rho U \\
&\qquad - A_\mu\partial_\nu UU^{-1}\partial_\rho UU^{-1} + \partial_\mu UU^{-1}\partial_\nu A_\rho\bigg) \\
&= \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\bigg(-\frac{1}{3}U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}\partial_\rho U - \partial_\mu\Big(\partial_\nu UU^{-1}A_\rho\Big)\bigg)
\end{aligned} \tag{59}
$$

The winding number of the group element $U \in G$ is

$$
w = \frac{1}{24\pi^2}\int d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big(U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}\partial_\rho U\Big) \in \mathbb{Z}, \qquad \pi_3(U(N)) \cong \mathbb{Z} \tag{60}
$$

Final we obtain the change of the action under the large gauge transformation

$$
S[A] \longrightarrow S[A^U] = S[A] - \kappa\,8\pi^2\,w \tag{61}
$$

In path integral quantization formalism, one needs the measure

$$[\mathcal{D}A]\, e^{iS[A]} \tag{62}$$

invariant under the gauge transformation. Firstly notice that the Lebesgue measure $\mathcal{D}A$ is gauge invariant:

$$[\mathcal{D}A] \equiv \prod_{a,\mu,x} dA^a_\mu(x) \qquad \Longrightarrow \qquad [\mathcal{D}A^U] = [\mathcal{D}A] \tag{63}$$

So we must require

$$S[A] \longrightarrow S[A^U] = S[A] - \kappa\, 8\pi^2\, w = S[A] + 2\pi N, \qquad N \in \mathbb{N} \tag{64}$$

This gives

$$\kappa = \frac{k}{4\pi}, \qquad S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int \mathrm{Tr}\left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad k \in \mathbb{N}^+ \tag{65}$$

Here the positive integer $k$ is called the Chern-Simons level.

As pointed out by Witten, consistency of quantum field theory does not quite require the single-valuedness of $S$, but only of $\exp(iS)$ [10].

## Topological invariants

Since the action of Chern-Simons theory

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right) \tag{66}$$

does not involve the metric, the resulting quantum theory is topological, at least formally. That is, the theory contains topological invariants. In particular, the partition function

$$Z(M) = \int [\mathcal{D}A]\, e^{iS[A]} \tag{67}$$

should define a topological invariant of the manifold $M$ without boundary. A detailed analysis shows that this is in fact the case, with an extra subtlety: the invariant depends not only on the three-manifold but also on a choice of framing [10, 11].

The observables of Chern-Simons theory are the $n$-point correlation functions of gauge-invariant operators. The most often studied class of gauge invariant operators are Wilson loops. A Wilson loop is the holonomy around a loop in $M$, traced in a given representation $R$ of $G$. More concretely, given an irreducible representation $R$ and a loop $\gamma$ in $M$, one may define the Wilson loop $W_R(\gamma)$ by

$$W_R(\gamma) \equiv \mathrm{Tr}_R\,\mathcal{P}\,\exp\left(i\oint_\gamma A\right) \tag{68}$$

where $\mathcal{P}$ is the path-ordered.

# Summary

The action of Chern-Simons theory in $D = 3$ dimensions is given by

$$S_{\text{CS}} = \frac{k}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{69}$$

- **Topological origin** The Chern-Simons form is related to a topological density in $2n$ dimensions known as a characteristic class $Q_{2n-1}$, through

$$P_{2n}(F) = dQ_{2n-1}(A, F) \tag{70}$$

$$\text{Tr} \left( F \wedge F \right) = d\, \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{71}$$

- **No metric required** As noticed by Witten [10], since this action does not involve the metric, the resulting quantum theory is topological, at least formally.
- **Orientation-preserving diffeomorphisms**
- **Gauge invariance** The Chern-Simons action is invariant under the infinitesimal gauge transformation. Under the large gauge transformation, the change of the action

$$S[A] \longrightarrow S[A^U] = S[A] - 2\pi\, k\, N, \qquad k \in \mathbb{N}^+ \tag{72}$$

The positive integer $k$ is called the level of the quantum Chern-Simons theory.

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