Quantization of the Bosonic String Theory

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Outline

1 The Classical Theory

The string action and Symmetries

We start from the Polyakov action of the string

$$S = -\frac{T}{2}\int d^2\sigma\,\sqrt{-h}\,h^{ab}\,\partial_a X^\mu \partial_b X_\mu \tag{1}$$

This action is local in X^μ and has rich symmetries.

D-dimensional Poincaré invariance

$$\sigma \longrightarrow \tilde{\sigma} = \sigma, \tag{2}$$

$$h_{ab} \longrightarrow \tilde{h}_{ab} = h_{ab}, \tag{3}$$

$$X^\mu \longrightarrow \tilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu. \tag{4}$$

Reparametrization invariance or Diff invariance:

$$\sigma^a \longrightarrow \tilde{\sigma}^a = \tilde{\sigma}^a(\sigma), \tag{5}$$

$$\tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma), \tag{6}$$

$$\tilde{h}_{ab}(\tilde{\sigma}) = \frac{\partial\sigma^c}{\partial\tilde{\sigma}^a}\frac{\partial\sigma^d}{\partial\tilde{\sigma}^b}\,h_{cd}(\sigma). \tag{7}$$

In infinitesimal form we have

$$\begin{aligned} \tilde{\sigma}^a &= \sigma^a - \xi^a, && (8)\\ \delta X^\mu &\equiv \tilde{X}^\mu(\sigma) - X^\mu(\sigma) = \xi^a\,\partial_a X^\mu, && (9)\\ \delta h^{ab} &\equiv \tilde{h}^{ab}(\sigma) - h^{ab}(\sigma) = \xi^c\,\partial_c h^{ab} - h^{ac}\partial_c\xi^b - h^{bc}\partial_c\xi^a \\ &= -\nabla^a\xi^b - \nabla^b\xi^a. && (10) \end{aligned}$$

Weyl scaling

$$\begin{aligned} \tilde{X}^\mu &= X^\mu, && (11)\\ \tilde{h}^{ab} &= e^{\phi(\sigma)}\,h^{ab}. && (12) \end{aligned}$$

Poincaré transformations are global symmetries, whereas reparametrizations and Weyl transformations are local symmetries. The local symmetries (Diff×Weyl) can be used to choose a gauge. It is a delicate problem to preserve all these symmetries in the process of quantization.

Equation of Motion (EOM)

$$\frac{\delta S}{\delta h^{ab}} = 0 \qquad \Longrightarrow \qquad T_{ab} \equiv -\frac{T}{2}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{ab}} = 0 \tag{13}$$

The equation of motion for h^{ab} implies the vanishing of the world-sheet energy-momentum tensor. This is a strong constraint.

Using the following formula:

$$\delta h \;=\; -\,h h_{ab}\,\delta h^{ab} \tag{14}$$

We easily obtain

$$T_{ab} \;=\; \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab}\, h^{cd}\,\partial_c X^\mu \partial_d X_\mu \;=\; 0 \tag{15}$$

The equation of motion for X^μ:

$$\Delta X^\mu \;=\; -\frac{1}{\sqrt{-h}}\partial_a\Big(\sqrt{-h}h^{ab}\partial_b X^\mu\Big) \;=\; 0 \tag{16}$$

Gauge Fixing

Theorem: Every 2-dimensional Riemannian manifold is locally conformally flat [16, 17, 8].

We'll work in Euclidean space to avoid annoying minus signs. Consider two metrics related by a Weyl transformation, $\tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}$. One can check that the Ricci scalars of the two metrics are related by,

$$\sqrt{\tilde{g}}\,\tilde{R} \;=\; \sqrt{g}\,\big(R - 2\nabla^2\phi\big) \tag{17}$$

We can therefore pick a ϕ such that the new metric has vanishing Ricci scalar, $\tilde{R} = 0$, simply by solving this differential equation for ϕ: $\nabla^2\phi = R/2$. In two dimensions there exists a relation between Riemann tensor and Ricci scalar

$$R_{\mu\nu\sigma\rho} = \frac{1}{2}R\left(g_{\mu\sigma}g_{\nu\rho} - g_{\nu\sigma}g_{\mu\rho}\right) \tag{18}$$

(Notice that this is generally not true in higher dimensions). So we have

$$\tilde{R} = 0 \qquad \Longrightarrow \qquad \tilde{R}_{\mu\nu\sigma\rho} = 0 \tag{19}$$

which means that the manifold is flat. Therefore, in two dimensions a vanishing Ricci scalar implies a flat metric. So we can further used reparameterization invariance to pick coordinates in which the flatness of the manifold is manifest.

Suppose we start from the general form of the metric

$$ds^2 = g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2 \tag{20}$$

Performing the following coordinate transformation

$$\frac{1}{\lambda}(du + idv) = \sqrt{g_{xx}}dx + \frac{g_{xy} + i\sqrt{\det(g_{ab})}}{\sqrt{g_{xx}}}dy \tag{21}$$

$$\frac{1}{\bar{\lambda}}(du - idv) = \sqrt{g_{xx}}dx + \frac{g_{xy} - i\sqrt{\det(g_{ab})}}{\sqrt{g_{xx}}}dy \tag{22}$$

$\Longrightarrow$

$$ds^2 = |\lambda|^{-2}\left(du^2 + dv^2\right) = e^{2\omega}\left(du^2 + dv^2\right), \qquad |\lambda|^{-2} = e^{2\omega} \tag{23}$$

Therefore, by using the general coordinate transformation, we can bring the world-sheet metric h_{ab} to the following form (locally):

$$(h_{ab}) = e^{\rho} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = e^{\rho}(\eta_{ab}) \tag{24}$$

This gauge is called the **conformal gauge**.

Using Weyl symmetry, we can further reduce the metric h_{ab} to a more simple form

$$h_{ab} = \eta_{ab} \tag{25}$$

Actually such a at world-sheet metric is only possible if there is no topological obstruction. This is the case when the world sheet has vanishing Euler characteristic. Examples include a cylinder and a torus. When a flat world-sheet metric is an allowed gauge choice, the string action takes the simple form

$$\begin{aligned} S &= -\frac{T}{2}\int d^2\sigma \sqrt{-h}\, h^{ab}\,\partial_a X^{\mu}\partial_b X_{\mu} \\ &= -\frac{T}{2}\int d^2\sigma\, \eta^{ab}\,\partial_a X^{\mu}\partial_b X_{\mu} \\ &= \frac{T}{2}\int d^2\sigma \left(\dot{X}^2 - X'^2\right) \end{aligned} \tag{26}$$

Thus the equation of motion for X^μ becomes

$$\eta^{ab}\partial_a\partial_b X^\mu = 0 \qquad \text{or} \qquad \left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = -\Box X^\mu = 0 \tag{27}$$

This is just the usual wave equation in 2 dimensions. A general solution is

$$X^\mu = f^\mu(\tau - \sigma) + g^\mu(\tau + \sigma) \tag{28}$$

On the other hand, the equation of motion for h_{ab} (15) becomes the constraints:

$$\begin{aligned} 0 = T_{ab} &= \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu \\ &= \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \eta_{ab} \eta^{cd} \partial_c X^\mu \partial_d X_\mu \end{aligned} \tag{29}$$

$\Longrightarrow$

$$T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0 \qquad \text{and} \qquad T_{01} = T_{10} = \dot{X} \cdot X' = 0\,. \tag{30}$$

Which can also be written as

$$\left(\dot{X} \pm X'\right)^2 = 0 \tag{31}$$

These are known as the **Virasoro constraints**. This form will be useful in light-cone gauge quantization.

Boundary Conditions

For closed string we should impose the **periodic condition**

$$X^\mu(\sigma+\pi,\tau) \;=\; X^\mu(\sigma,\tau) \tag{32}$$

For convenience, we have choosen the coordinate σ to have the range $\sigma \in [0,\,\pi]$. For closed string living on compact space, this periodic condition can be relaxed and the correct condition for the compact dimension X^m is

$$X^m(\sigma+\pi,\tau) = X^m(\sigma,\tau) \;+\; 2\,\pi\,R\,w^m, \tag{33}$$

where R is the radius of the compact space and w^m is an integer. We note that closed strings on compact space have more degrees of freedom which are described by the winding number w^m.

For open string, we should return back to the action (26) and find the correct boundary condition.

$$\frac{\delta S}{\delta X^\mu} = 0 \qquad \Longrightarrow \qquad X'_\mu \delta X^\mu\big|_{\sigma=\pi} - X'_\mu \delta X^\mu\big|_{\sigma=0} = 0 \tag{34}$$

It is a reasonable assumption that the two ends of the open string should be independent and so the two terms in the above equation is independent and should vanish independently. We thus arrive the following boundary condition

$$\partial_\sigma X^\mu = 0 \quad \text{or} \quad \delta X^\mu = 0, \quad \text{at } \sigma = 0,\,\pi \tag{35}$$

- The first boundary condition is called the **Numann boundary condition**:

$$\partial_\sigma X^\mu = 0 \quad \text{at } \sigma = 0,\ \pi \tag{36}$$

In this case the component of the momentum normal to the boundary of the world sheet vanishes. Physically, they mean that no momentum is flowing through the ends of the string.

- The second boundary condition is called the **Dirichlet boundary condition**:

$$X^\mu\Big|_{\sigma=0} = X_0^\mu, \qquad \text{and} \quad X^\mu\Big|_{\sigma=\pi} = X_\pi^\mu \tag{37}$$

where X_0^μ and X_π^μ are constants. In this case the positions of the two string ends are fixed. The modern interpretation is that X_0^μ and X_π^μ represent the positions of Dp-branes.

Closed/Open string mode expansions

World-sheet light-cone coordinates

$$\sigma^{\pm} = \tau \pm \sigma \tag{38}$$

In these coordinates the derivatives and the two-dimensional Lorentz metric take the form

$$\partial_{\pm} \equiv \frac{\partial}{\partial\sigma^{\pm}} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \qquad \text{and} \qquad \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{39}$$

The EOM for X^μ (27) becomes

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = 0 \qquad \Longrightarrow \qquad \partial_+\partial_- X^\mu = 0 \tag{40}$$

The vanishing of the energy-momentum tensor becomes

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0, \tag{41}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu = 0, \tag{42}$$

$$T_{+-} = T_{-+} = 0, \tag{43}$$

Those components can obtain by the transformation relation of the tensor

$$\tilde{T}_{\alpha\beta} = \frac{\partial\sigma^a}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^b}{\partial\tilde{\sigma}^\beta} T_{ab}, \qquad \sigma = (\tau, \sigma), \quad \tilde{\sigma} = (\sigma^+, \sigma^-) \tag{44}$$

The general solution of the wave equation (40) is given by

$$X^\mu(\tau,\sigma) \;=\; X^\mu_{\rm R}(\tau-\sigma) + X^\mu_{\rm L}(\tau+\sigma) \tag{45}$$

which is a sum of **right-movers** and **left-movers**. To find the explicit form of $X^\mu_{\rm R}$ and $X^\mu_{\rm L}$ one should require X^μ to be real and impose the constraints

$$\big(\partial_- X_{\rm R}\big)^2 \;=\; \big(\partial_+ X_{\rm L}\big)^2 \;=\; 0 \tag{46}$$

Closed-string mode expansion

Expanding the fuction $X^\mu_{\rm R}$ and $X^\mu_{\rm L}$ into Fourier series, we can obtain the most general solution of the wave equation satisfying the closed-string boundary condition

$$X^\mu_{\rm R} \;=\; \frac{1}{2}x^\mu + \frac{1}{2}\ell_s^2\, p^\mu(\tau-\sigma) + \frac{i}{2}\ell_s \sum_{n\neq 0} \frac{\alpha^\mu_n}{n}\, e^{-2in(\tau-\sigma)}, \tag{47}$$

$$X^\mu_{\rm L} \;=\; \frac{1}{2}x^\mu + \frac{1}{2}\ell_s^2\, p^\mu(\tau+\sigma) + \frac{i}{2}\ell_s \sum_{n\neq 0} \frac{\tilde\alpha^\mu_n}{n}\, e^{-2in(\tau+\sigma)} \tag{48}$$

where x^μ is a **center-of-mass position** and p^μ is the **total string momentum**, describing the free motion of the string center of mass. The exponential terms represent the string excitation modes.

The requirement that $X^\mu_{\rm R}$ and $X^\mu_{\rm L}$ are real functions implies that x^μ and p^μ are real, while positive and negative modes are conjugate to each other

$$\alpha_{-n} = (\alpha_n)^* \quad \text{and} \quad \tilde{\alpha}_{-n} = (\tilde{\alpha}_n)^* \tag{49}$$

Here we have introduced a new parameter, the **string length scale ℓ_s**, which is related to the string tension T and the open-string **Regge slope** parameter α' by

$$T = \frac{1}{2\pi\alpha'} \qquad \text{and} \qquad \frac{1}{2}\ell_s^2 = \alpha' \quad \Longrightarrow \quad \ell_s = \sqrt{2\alpha'} \tag{50}$$

An episode of α' [6]

If we consider a rigidly rotating open string, we can easily conclude that the angular momentum of the string is proportional to the square of its energy. More explicitly

$$J \;=\; \alpha' E^2 \tag{51}$$

α' is the proportionality constant. The name slope parameter arises because α' is the slope of the lines of J, when plotted as a function of energy-squared. In fact, Regge trajectories are approximate lines that arise when plotting angular momentum as a function of energy-squared for hadronic excitations. In the early 1970s, when string theory was investigated as a theory of strong interactions, the slope parameter α' was the experimentally measured quantity that entered into the string action.

For closed string when we consider the case of compact dimensions, this periodic condition can be replaced for the compact dimension X^m [10, 12]

$$X^m(\sigma + \pi, \tau) = X^m(\sigma, \tau) + 2\,\pi\,R\,w^m, \tag{52}$$

In this case the mode expansion for closed string is

$$X^\mu(\tau,\sigma) \;=\; x^\mu + \alpha'\,p^\mu\,\tau + R\,w^\mu\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left[\frac{\alpha_n^\mu}{n}\,e^{-2i\,n(\tau-\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n}\,e^{-2i\,n(\tau+\sigma)}\right]. \tag{53}$$

For non-compact dimension, we should set ($R \to \infty$ and) $w^\mu \to 0$ (to get a finite result).

We calculate several useful expressions

$$X^\mu(\tau,\sigma) \;=\; x^\mu + i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau \;+\; i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left[\frac{\alpha_n^\mu}{n}\,e^{-2in(\tau-\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n}\,e^{-2in(\tau+\sigma)}\right] \tag{54}$$

$$\partial_- X_{\mathrm{R}}^\mu \;=\; \sqrt{2\alpha'}\sum_{n=-\infty}^{n=+\infty}\alpha_n^\mu\,e^{-2in(\tau-\sigma)} \qquad \partial_+ X_{\mathrm{L}}^\mu \;=\; \sqrt{2\alpha'}\sum_{n=-\infty}^{n=+\infty}\tilde{\alpha}_n^\mu\,e^{-2in(\tau+\sigma)} \tag{55}$$

$$\dot{X}^\mu(\tau,\sigma) \;=\; \sqrt{2\alpha'}\sum_{n=-\infty}^{n=+\infty}\left[\alpha_n^\mu\,e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu\,e^{-2in(\tau+\sigma)}\right], \qquad \alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}\,p^\mu \tag{56}$$

$$X'^\mu(\tau,\sigma) \;=\; \sqrt{2\alpha'}\sum_{n=-\infty}^{n=+\infty}\left[-\alpha_n^\mu\,e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu\,e^{-2in(\tau+\sigma)}\right] \tag{57}$$

The canonical momentum conjugate to X^μ is defined as

$$P^\mu(\tau,\sigma) \;\equiv\; \frac{\delta S}{\delta \dot{X}_\mu} \;=\; T\dot{X}^\mu \;=\; \frac{1}{\pi}\frac{1}{2\alpha'}\sum_{n=-\infty}^{n=+\infty}\left[\alpha_n^\mu\, e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu\, e^{-2in(\tau+\sigma)}\right] \tag{58}$$

The classical Poisson brackets are

$$\big[X^\mu(\sigma,\tau),\, X^\nu(\sigma',\tau)\big]_{\text{P.B.}} \;=\; \big[P^\mu(\sigma,\tau),\, P^\nu(\sigma',\tau)\big]_{\text{P.B.}} \;=\; 0 \tag{59}$$

$$\big[X^\mu(\sigma,\tau),\, P^\nu(\sigma',\tau)\big]_{\text{P.B.}} \;=\; \eta^{\mu\nu}\,\delta(\sigma'-\sigma) \tag{60}$$

or in terms of $\dot{X}^\mu$

$$\big[X^\mu(\sigma,\tau),\, \dot{X}^\nu(\sigma',\tau)\big]_{\text{P.B.}} \;=\; T^{-1}\,\eta^{\mu\nu}\,\delta(\sigma'-\sigma) \tag{61}$$

Inserting the mode expansion for X^μ (54) and $\dot{X}^\mu$ (56) into these equations gives the Poisson brackets satisfied by the modes

$$\big[x^\mu,\, p^\nu\big]_{\text{P.B.}} \;=\; \eta^{\mu\nu} \tag{62}$$

$$\big[\alpha_m^\mu,\, \alpha_n^\nu\big]_{\text{P.B.}} \;=\; \big[\tilde{\alpha}_m^\mu,\, \tilde{\alpha}_n^\nu\big]_{\text{P.B.}} \;=\; -\,im\eta^{\mu\nu}\delta_{m+n,0} \tag{63}$$

$$\big[\alpha_m^\mu,\, \tilde{\alpha}_n^\nu\big]_{\text{P.B.}} \;=\; 0 \tag{64}$$

Where we have used the Fourier expansion of the δ-fuction

$$\delta(\sigma-\sigma') \;=\; \frac{1}{\pi}\sum_{n=-\infty}^{n=+\infty} e^{2in(\sigma-\sigma')}, \qquad \text{for close string} \tag{65}$$

Open-string mode expansion

For Numann boundary codition, the mode expansion for the open string is

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \tag{66}$$

$$= x^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \tag{67}$$

Where

$$\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu \tag{68}$$

For details of the derivation, see Barton Zwiebach 2009 [6], *A First Course in String Theory*, section 9.4. We will not discuss the Dirichlet bounday condition here and note that it will be important in discussing D-branes.

For open string we have the following mode expansions

$$\mathcal{P}^\mu(\tau,\sigma) = T\dot{X}^\mu = T\sqrt{2\alpha'} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) = \frac{1}{\pi}\frac{1}{\sqrt{2\alpha'}} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \tag{69}$$

$$X'_\mu(\tau,\sigma) = -i\sqrt{2\alpha'} \sum_{n\neq 0} \alpha_n^\mu e^{-in\tau} \sin(n\sigma) \tag{70}$$

$$2\partial_\pm X^\mu = \dot{X}^\mu \pm X'^\mu = \sqrt{2\alpha'} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu e^{-in(\tau\pm\sigma)} \tag{71}$$

Using the classical Poisson brackets

$$\left[X^\mu(\sigma,\tau),\, X^\nu(\sigma',\tau)\right]_{\text{P.B.}} \;=\; \left[P^\mu(\sigma,\tau),\, P^\nu(\sigma',\tau)\right]_{\text{P.B.}} \;=\; 0 \tag{72}$$

$$\left[X^\mu(\sigma,\tau),\, P^\nu(\sigma',\tau)\right]_{\text{P.B.}} \;=\; \eta^{\mu\nu}\,\delta(\sigma'-\sigma) \tag{73}$$

and the Fourier expansion of the δ-fuction

$$\delta(\sigma-\sigma') \;=\; \frac{1}{\pi}\sum_{n\in\mathbb{Z}}^{\infty}\cos(n\sigma)\,\cos(n\sigma'), \tag{74}$$

we can obtain the Poisson brackets satisfied by the modes in case of the open string

$$\left[x^\mu,\, p^\nu\right]_{\text{P.B.}} \;=\; \eta^{\mu\nu} \tag{75}$$

$$\left[\alpha^\mu_m,\, \alpha^\nu_n\right]_{\text{P.B.}} \;=\; -\,im\eta^{\mu\nu}\delta_{m+n,0} \tag{76}$$

Hamiltonian and energy-momentum tensor

Noether theorem

For a global symmetry transformation

$$\phi \longrightarrow \phi + \delta_\epsilon \phi \tag{77}$$

Such a transformation is a symmetry of the theory if it leaves the equations of motion invariant.

This is the case if the action changes at most by a surface term, which means that the Lagrangian density changes at most by a total derivative. The Noether current is then determined from the change in the action under the above transformation

$$\mathcal{L} \longrightarrow \mathcal{L} + \epsilon \partial_a \mathcal{J}^a \tag{78}$$

Where ϵ is a overall constant which reflects the fact that there is a global symmetry.

The Poincaré transformations

$$\delta X^\mu = \omega^\mu{}_\nu x^\nu + \epsilon^\mu \tag{79}$$

are global symmetries of the string world-sheet theory. Therefore, they give rise to conserved Noether currents.

- **Translation**, $\delta X^\mu = \epsilon^\mu$

$$\begin{aligned} \delta_\epsilon S &= \int d^2\sigma \left(\frac{\delta \mathcal{L}}{\delta X_\mu} \delta X_\mu + \frac{\delta \mathcal{L}}{\delta(\partial^a X_\mu)} \delta(\partial^a X_\mu) \right) \\ &= \int d^2\sigma \left(\frac{\delta \mathcal{L}}{\delta X_\mu} \delta X_\mu - \partial^a \frac{\delta \mathcal{L}}{\delta(\partial^a X_\mu)} \delta X_\mu + \partial^a \left(\frac{\delta \mathcal{L}}{\delta(\partial^a X_\mu)} \delta X_\mu \right) \right) \end{aligned} \tag{80}$$

Since

$$\mathcal{L} = -\frac{T}{2} \eta^{ab} \partial_a X^\mu \partial_b X_\mu \qquad \Longrightarrow \qquad \frac{\delta \mathcal{L}}{\delta(\partial^a X_\mu)} = -T \partial_a X^\mu \tag{81}$$

We obtain the Noether current

$$\mathcal{P}_a^\mu = T \partial_a X^\mu \tag{82}$$

Corresponding conserved charge is

$$P^\mu = \int_0^\pi d\sigma \mathcal{P}_0^\mu = T \int_0^\pi d\sigma \dot{X}^\mu = \left\{ \begin{array}{ll} \pi T \sqrt{2\alpha'} \alpha_0^\mu & \text{open string} \\ \pi T \sqrt{2\alpha'} \left(\alpha_0^\mu + \tilde{\alpha}_0^\mu \right) & \text{closed string} \end{array} \right\} = p^\mu \tag{83}$$

so that the total momentum of the string is the same as the 'momentum' p^μ of the zero mode.

- **Lorentz transformation**, $\delta X^\mu = \omega^\mu{}_\nu x^\nu$

$$\mathcal{J}_a^{\mu\nu} = T \left(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu \right) \tag{84}$$

The conserved charge

$$J^{\mu\nu} \;=\; \int_0^\pi d\sigma \mathcal{J}_0^{\mu\nu} \;=\; T\int_0^\pi d\sigma\big(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\big) \tag{85}$$

Inserting the mode expansions one obtains

$$\textbf{open string:}\;\; J^{\mu\nu} \;=\; l^{\mu\nu} \;+\; E^{\mu\nu} \qquad \textbf{closed string:}\;\; J^{\mu\nu} \;=\; l^{\mu\nu} \;+\; E^{\mu\nu} \;+\; \tilde{E}^{\mu\nu} \tag{86}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \quad E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\Big(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu\Big), \quad \tilde{E}^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\Big(\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu\Big) \tag{87}$$

Hamiltonian

$$H \;=\; \int_0^\pi d\sigma(\mathcal{P}_0^\mu\, \dot{X}_\mu - \mathcal{L}) \;=\; \frac{T}{2}\int_0^\pi d\sigma\big(\dot{X}^2 + X'^2\big) \tag{88}$$

Inserting the mode expansions for the closed-string

$$\dot{X}^\mu(\tau,\sigma) \;=\; \sqrt{2\alpha'}\sum_{n=-\infty}^{n=+\infty}\Big[\alpha_n^\mu\, e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu\, e^{-2in(\tau+\sigma)}\Big], \qquad \alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}\, p^\mu \tag{89}$$

$$X'^\mu(\tau,\sigma) \;=\; \sqrt{2\alpha'}\sum_{n=-\infty}^{n=+\infty}\Big[-\alpha_n^\mu\, e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu\, e^{-2in(\tau+\sigma)}\Big] \tag{90}$$

The result is

$$
\begin{aligned}
H &= \frac{T}{2}\int_0^\pi d\sigma\big(\dot{X}^2 + X'^2\big) \\
&= \frac{T}{2}\times(2\alpha')\times 2\int_0^\pi d\sigma\sum_{m,n}\Big(\alpha_m\cdot\alpha_n\, e^{-2i(m+n)(\tau-\sigma)} + \tilde{\alpha}_m\cdot\tilde{\alpha}_n\, e^{-2i(m+n)(\tau+\sigma)}\Big) \\
&= 2\alpha' T\sum_{m,n}\int_0^\pi d\sigma\Big(\alpha_m\cdot\alpha_n\, e^{-2i(m+n)(\tau-\sigma)} + \tilde{\alpha}_m\cdot\tilde{\alpha}_n\, e^{-2i(m+n)(\tau+\sigma)}\Big) \\
&= \sum_{n=-\infty}^{+\infty}\Big(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\Big)
\end{aligned}
\tag{91}
$$

In the third line we use the δ-function:

$$
\delta(x) = \frac{1}{2\pi}\int dp e^{-ipx} \qquad \Longrightarrow \qquad \int d\sigma e^{-2i(m+n)\sigma} = \pi\,\delta(m+n) \tag{92}
$$

For the open string the corresponding expression is

$$
H = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\alpha_{-n}\cdot\alpha_n \tag{93}
$$

Energy-momentum tensor

Let us recall Eq. (41) and Eq. (42)

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0$$
$$T_{--} = \partial_- X^\mu \partial_- X_\mu = 0$$

Inserting the mode expansions for the closed-string

$$\partial_- X_{\mathrm{R}}^\mu = \sqrt{2\alpha'} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu \, e^{-2in(\tau-\sigma)} \qquad \partial_+ X_{\mathrm{L}}^\mu = \sqrt{2\alpha'} \sum_{n=-\infty}^{n=+\infty} \tilde{\alpha}_n^\mu \, e^{-2in(\tau+\sigma)} \tag{94}$$

The result is

$$T_{--} = 4\alpha' \sum_{m=-\infty}^{+\infty} L_m \, e^{-2im(\tau-\sigma)} \quad \text{and} \quad T_{++} = 4\alpha' \sum_{m=-\infty}^{+\infty} \tilde{L}_m \, e^{-2im(\tau+\sigma)} \tag{95}$$

where the Fourier coefficients are the **Virasoro generators**

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{m-n} \cdot \alpha_n \quad \text{and} \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \tag{96}$$

In the same way, one can get the result for the modes of the energy-momentum tensor of the open string.

Comparing with the Hamiltonian, we find

$$H \;=\; \sum_{n=-\infty}^{+\infty} \Big(\alpha_{-n} \cdot \alpha_n \,+\, \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \Big) \;=\; 2\big(L_0 + \tilde{L}_0\big) \tag{97}$$

for a closed string, while for an open string

$$H \;=\; \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_n \;=\; L_0 \tag{98}$$

Mass formula for the string

Classically the vanishing of the energy-momentum tensor implies

$$T_{++} = 0 \quad \text{and} \quad T_{--} = 0 \qquad \Longrightarrow \qquad L_m = 0,\; m \in \mathbb{Z} \tag{99}$$

The classical constraint

$$L_0 = \tilde{L}_0 = 0 \tag{100}$$

can be used to derive an expression for the mass of a string. The relativistic mass-shell condition is

$$p^\mu p_\mu \;=\; -M^2 \tag{101}$$

where p^μ is the total momentum of the string

$$p^\mu = \int_0^\pi d\sigma P_0^\mu = T \int_0^\pi d\sigma \dot{X}^\mu \tag{102}$$

For the open string

$$L_0 = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n + \frac{1}{2}\alpha_0^2 = \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n + \alpha' p^2, \quad \alpha_0^\mu = \sqrt{2\alpha'} p^\mu \tag{103}$$

$\Longrightarrow$

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n \tag{104}$$

For closed string, using the same way, we obtain

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{+\infty} \left(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \right) \tag{105}$$

These are the mass-shell conditions for the string, which determine the mass of a given string state. In the quantum theory these relations get slightly modified.

2 Quantization – Old Covariant Approach

From Classical to Quantum

$$\frac{dA}{dt} = \big[A,\ H\big]_{\text{P.B.}} \qquad \Longrightarrow \qquad \frac{d\hat{A}}{dt} = \frac{1}{i\hbar}\big[\hat{A},\ \hat{H}\big] \tag{106}$$

$$\big[A(x,p),\ B(x,p)\big]_{\text{P.B.}} \qquad \Longrightarrow \qquad \frac{1}{i\hbar}\big[\hat{A}(x,p),\ \hat{B}(x,p)\big] \tag{107}$$

$$\big[x_i,\ p_j\big]_{\text{P.B.}} = \delta_{ij} \qquad \Longrightarrow \qquad \big[\hat{x}_i,\ \hat{p}_j\big] = i\hbar\,\delta_{ij} \tag{108}$$

According to above procedure of canonical quantization, we have

$$\big[x^\mu,\ p^\nu\big] \ = \ i\eta^{\mu\nu} \tag{109}$$

$$\big[\alpha^\mu_m,\ \alpha^\nu_n\big] \ = \ \big[\tilde{\alpha}^\mu_m,\ \tilde{\alpha}^\nu_n\big] \ = \ m\eta^{\mu\nu}\delta_{m+n,0} \tag{110}$$

and the rest commutation relations are 0.

Defining

$$a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m \qquad \text{and} \qquad a^{\mu\dagger}_m = \frac{1}{\sqrt{m}}\alpha^\mu_{-m} \qquad \text{for} \qquad m > 0 \tag{111}$$

In the case of closed strings, there are also corresponding $\tilde{\alpha}^\mu_m$. Then the algebra satisfied by the modes is essentially the algebra of raising and lowering operators for quantum-mechanical harmonic oscillators

$$\left[a_m^\mu,\, a_n^{\nu\dagger}\right] \;=\; \left[\tilde{a}_m^\mu,\, \tilde{a}_n^{\nu\dagger}\right] \;=\; \eta^{\mu\nu}\delta_{m,n} \qquad \text{for} \qquad m,n>0 \tag{112}$$

The spectrum is constructed by applying raising operators on the ground state, which is denoted $|0\rangle$

$$a_m^\mu \left|0\right\rangle = 0 \qquad \text{for} \qquad m>0 \tag{113}$$

One can also specify the momentum k^μ carried by a state $|\phi\rangle$,

$$\left|\phi\right\rangle \;\equiv\; a_{m_1}^{\mu_1} a_{m_2}^{\mu_2} \cdots a_{m_n}^{\mu_n} \left|0;k\right\rangle, \qquad p^\mu \left|\phi\right\rangle \;=\; k^\mu \left|\phi\right\rangle \tag{114}$$

There is just a unusual feature: the commutators of time components have a negative sign

$$\left[a_m^0,\, a_n^{0\dagger}\right] \;=\; \left[\tilde{a}_m^0,\, \tilde{a}_n^{0\dagger}\right] \;=\; -1 \tag{115}$$

This will be further discussed in the following.

The states with an even number of time-component operators have positive norm, while those that are constructed with an odd number of time component operators have negative norm. A simple example of a negative-norm state is given by

$$a_m^{0\dagger}\left|0\right\rangle, \qquad \text{with norm} \qquad \left\langle 0\right| a_m^0 a_m^{0\dagger} \left|0\right\rangle \;=\; -1 \qquad \langle 0|0\rangle = 1 \tag{116}$$

In order for the theory to be physically sensible, it is essential that all physical states have positive norm. Negative-norm states in the physical spectrum of an interacting theory would lead to violations of causality and unitarity. So we have to eliminate the negative-norm states from the physical spectrum.

The Witt algebra

In the classical theory the Virasoro generators satisfy the algebra

$$\left[L_m,\, L_n\right]_{\text{P.B.}} \;=\; -\,i(m-n)L_{m+n} \tag{117}$$

This is called **Witt algebra**, or classical Virasoro algebra. The appearance of the Witt algebra is due to the fact that the gauge choice $h_{ab} = \eta_{ab}$ has not fully fixed the reparametrization symmetry. Let us recall the local symmetries (Diff×Weyl) of the string action:

- **Reparametrization invariance**

$$\sigma^a \;\longrightarrow\; \tilde{\sigma}^a \;=\; \tilde{\sigma}^a(\sigma), \tag{118}$$

$$\tilde{X}^\mu(\tilde{\sigma}) \;=\; X^\mu(\sigma), \tag{119}$$

$$\tilde{h}_{ab}(\tilde{\sigma}) \;=\; \frac{\partial\sigma^c}{\partial\tilde{\sigma}^a}\frac{\partial\sigma^d}{\partial\tilde{\sigma}^b}\,h_{cd}(\sigma). \tag{120}$$

- **Weyl scaling**

$$\tilde{X}^\mu \;=\; X^\mu, \tag{121}$$

$$\tilde{h}^{ab} \;=\; e^{\phi(\sigma)}\,h^{ab}. \tag{122}$$

Consider a infinitesimal symmetric transformation

$$\sigma^a \;\longrightarrow\; \tilde{\sigma}^a \;=\; \sigma^a + \xi^a(\sigma) + \mathcal{O}(\xi^2)$$

Where ξ^a is an infinitesimal parameter for a reparametrization and let Λ be an infinitesimal parameter for a Weyl rescaling. We have

$$\begin{aligned} e^{\Lambda}\eta_{ab} &= \frac{\partial\sigma^c}{\partial\tilde{\sigma}^a}\frac{\partial\sigma^d}{\partial\tilde{\sigma}^b}\eta_{cd} \\ \big(1+\Lambda+\mathcal{O}(\Lambda^2)\big)\eta_{ab} &= \Big(\delta_a^c+\partial_a\xi^c+\mathcal{O}(\xi^2)\Big)\Big(\delta_b^d+\partial_b\xi^d+\mathcal{O}(\xi^2)\Big)\eta_{cd} \\ &= \eta_{ab}+\partial_a\xi_b+\partial_b\xi_a+\mathcal{O}(\xi^2) \end{aligned} \tag{123}$$

$\Longrightarrow$

$$\partial^a\xi^b+\partial^b\xi^a = \Lambda\eta^{ab} \tag{124}$$

Obviously, this residual symmetry respects the previous gauge choice $h_{ab}=\eta_{ab}$. This is just **conformal symmetry**, we will further discuss it in the next chapter.

Defines the combinations

$$\xi^{\pm} = \xi^0\pm\xi^1, \qquad \sigma^{\pm} = \sigma^0\pm\sigma^1 \tag{125}$$

then we can find that Eq. (124) is solved by

$$\xi^{+} = \xi^{+}(\sigma^{+}), \qquad \xi^{-} = \xi^{-}(\sigma^{-}) \tag{126}$$

This plays a important role in light-cone gauge quatization. The infinitesimal generators for the transformations $\delta\sigma^{\pm} = \xi^{\pm}$ are given by

$$V^{\pm} \;=\; \frac{1}{2}\xi^{\pm}(\sigma^{\pm})\frac{\partial}{\partial\sigma^{\pm}} \;=\; \frac{1}{2}\xi^{\pm}(\sigma^{\pm})\partial_{\pm} \tag{127}$$

and a complete basis for these transformations is given by

$$\xi_n^{\pm}(\sigma^{\pm}) \;=\; e^{2in\sigma^{\pm}}, \qquad n \in \mathbb{Z} \tag{128}$$

The corresponding generators $V_n^{\pm}$ give two copies of the Virasoro algebra. In the case of open strings there is just one Virasoro algebra, and the infinitesimal generators are

$$V_n \;=\; e^{in\sigma^+}\partial_+ + e^{in\sigma^-}\partial_- \qquad n \in \mathbb{Z} \tag{129}$$

Next we show they obey the Witt algebra (117)

$$\begin{aligned}
\left[V_m,\, V_n\right] \;&=\; \left[e^{im\sigma^+}\partial_+ + e^{im\sigma^-}\partial_-,\, e^{in\sigma^+}\partial_+ + e^{in\sigma^-}\partial_-\right] \\
&=\; -i(m-n)\left(e^{i(m+n)\sigma^+}\partial_+ + e^{i(m+n)\sigma^-}\partial_-\right) \\
&=\; -i(m-n)V_{m+n}
\end{aligned} \tag{130}$$

The Virasoro Algebra

Now we have to face a typical **ambiguity** when quantizing a classical system. The classical variables are functions of coordinates and momenta. However, in the quantum theory, coordinates and momenta are non-commuting operators. A specific ordering prescription has to be made in order to define them as operators in the quantum theory. In particular, Virasoro operators are defined by their normal-ordering in the quantum theory

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} :\alpha_{m-n}\cdot\alpha_n: \qquad \left[\alpha^\mu_{m-n},\, \alpha^\nu_n\right] \sim \delta_{m,0} \tag{131}$$

The normal-ordering is defined as

$$:\alpha^\mu_m\alpha^\nu_n: \;\equiv \begin{cases} \alpha^\mu_m\alpha^\nu_n & \text{if} \quad m<n \\ \alpha^\nu_n\alpha^\mu_m & \text{if} \quad m>n \end{cases} \tag{132}$$

Then all L_m for $m \neq 0$ are all fine promoted to quantum operator. However, L_0 becomes

$$L_0 = \frac{1}{2}\sum_{n=-\infty}^{+\infty} :\alpha_{-n}\cdot\alpha_n: \; = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{+\infty}\alpha_{-n}\cdot\alpha_n \tag{133}$$

Since an arbitrary constant could have appeared in this expression, we must add a constant to L_0 in all formulas. Otherwise stated, we replace L_0 by $L_0 - a$.

Let us observe the classical version of the Virasoro algebra (Witt algebra)

$$\big[L_m,\, L_n\big]_{\text{P.B.}} \;=\; -\,i(m-n)L_{m+n} \tag{134}$$

On the right hand side of the expression, any normal-ordering ambiguity will arise for $m+n=0$, so we need add a c-number in Quantum version:

$$\big[L_m,\, L_n\big] \;=\; (m-n)L_{m+n} \,+\, A(m)\delta_{m+n,0} \tag{135}$$

By studying the Jacobi identity

$$[L_k,\,[L_n,\,L_m]] \,+\, [L_n,\,[L_m,\,L_k]] \,+\, [L_m,\,[L_k,\,L_n]] \;=\; 0 \tag{136}$$

When $k+n+m=0$, we have

$$(n-m)A(k) \,+\, (m-k)A(n) \,+\, (k-n)A(m) \;=\; 0 \tag{137}$$

Setting $k=1$ and $m=-n-1$, one obtains

$$A(n+1) \;=\; \frac{(n+2)A(n) \,-\, (2n+1)A(1)}{n-1} \tag{138}$$

This recursion relation is enough to determine all of the $A(n)$ in terms of $A(1)$ and $A(2)$. In fact, a general solution is

$$A(m) \;=\; c_1 m \,+\, c_3 m^3 \tag{139}$$

In order to determine above two unknown coefficients, we consider

$$\langle 0;0|\,[L_1,\ L_{-1}]\,|0;0\rangle \qquad \text{and} \qquad \langle 0;0|\,[L_2,\ L_{-2}]\,|0;0\rangle\,. \tag{140}$$

- First we substitute the mode expansions of the $L_{\pm 1}$ into $\langle 0;0|\,[L_1,\ L_{-1}]\,|0;0\rangle$

$$\langle 0;0|\,[L_1,\ L_{-1}]\,|0;0\rangle \;=\; 0 \tag{141}$$

On the other hand, using the extension of the Varasoro algebra (135), have

$$\langle 0;0|\,[L_1,\ L_{-1}]\,|0;0\rangle \;=\; c_3 + c_1 \qquad\qquad (p^\mu\,|0;0\rangle = 0) \tag{142}$$

- Similarly, substitute the mode expansions of the $L_{\pm 2}$ into $\langle 0;0|\,[L_2,\ L_{-2}]\,|0;0\rangle$,

$$\begin{aligned}
\langle 0;0|\,[L_2,\ L_{-2}]\,|0;0\rangle &= \langle 0;0|\,[L_2 L_{-2}]\,|0;0\rangle \\
&= \frac{1}{4}\,\langle 0;0|\,\alpha_1\cdot\alpha_1\,\alpha_{-1}\cdot\alpha_{-1}\,|0;0\rangle \\
&= \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\lambda}\,\langle 0;0|\,\alpha_1^\mu\alpha_1^\nu\alpha_{-1}^\rho\alpha_{-1}^\lambda\,|0;0\rangle \\
&= \frac{1}{2}\eta_{\mu\nu}\,\langle 0;0|\,\alpha_1^\mu\alpha_{-1}^\nu\,|0;0\rangle \\
&= \frac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu} \;=\; \frac{D}{2}
\end{aligned} \tag{143}$$

On the other hand, we have

$$\langle 0;0|\,[L_2,\ L_{-2}]\,|0;0\rangle \;=\; \langle 0;0|\,[L_2 L_{-2}]\,|0;0\rangle \;=\; 8c_3 \,+\, 2c_1 \tag{144}$$

From above

$$c_1 + c_3 = 0 \quad \text{and} \quad 2c_2 + 8c_3 = \frac{D}{2} \qquad \Longrightarrow \qquad c_3 = \frac{D}{12} = -c_1 \tag{145}$$

Therefore we obtain the Virasoro algebra in the quantum theory

$$\boxed{\left[L_m,\, L_n\right] \;=\; (m-n)L_{m+n} \;+\; \frac{c}{12}m(m^2-1)\delta_{m+n,0}} \tag{146}$$

where $c = D$ is the target space (space-time) dimension. The term proportional to c is a quantum effect. This means that it appears after quantization and is absent in the classical theory. This term is called a central extension, and c is called a central charge, since it can be regarded as multiplying the unit operator, which when adjoined to the algebra is in the center of the extended algebra.

Notice that $A(0) = A(\pm 1) = 0$, it is easy to verify L_0 and $L_{\pm 1}$ generator a closed subalgebra of the Varasoro algebra, without anomaly

$$\left[L_{\pm 1},\, L_0\right] \;=\; \pm L_{\pm 1}, \quad \left[L_1,\, L_{-1}\right] \;=\; L_0 \tag{147}$$

This is just $sl(2,\mathbb{R})$ or $su(1,1)$. In the case of the closed string, the complete Virasoro algebra of both left-movers and right-movers contains the subalgebra $sl(2,\mathbb{R}) \times sl(2,\mathbb{R}) = so(2,2)$. This subalgebra plays a important role in string theory.

Physical states

In the classical theory, a important example of the constraints is the condition that L_0 must vanish $(L_0 = 0)$ for the allowed motions of the string. The most naive quantum mechanical analog of that requirement would be the statement that L_0 should annihilate physical states:

$$L_0 |\phi\rangle = 0 \qquad \Longrightarrow \qquad 0 = \langle\phi|[L_m, L_{-m}]|\phi\rangle = \frac{c}{12}m(m^2-1) \neq 0 \ \text{✗} \tag{148}$$

Where $|\phi\rangle = |\texttt{phys.}\rangle$ is any physical on-shell state in the theory. However, in the quantum theory a constant may need to be added to L_0 to parametrize the arbitrariness in the ordering prescription. Therefore, when imposing the constraint that the zero mode of the energy-momentum tensor should vanish, the only requirement in the case of the open string is that there exists some constant a such that

$$\textbf{Classical:} \quad L_0 = 0 \qquad \text{vs} \qquad \textbf{Quantum:} \quad \big(L_0 - a\big)|\phi\rangle = 0 \tag{149}$$

The constant a will be determined later. Similarly, for the closed string

$$\big(L_0 - a\big)|\phi\rangle = \big(\tilde{L}_0 - a\big)|\phi\rangle = 0 \tag{150}$$

In quantum theory one cannot demand that the operator L_m annihilates all the physical states, for all $m \neq 0$, since this is incompatible with the Virasoro algebra.

$$\left[L_m,\, L_{-m}\right] \;=\; 2mL_0 + \frac{c}{12}m(m^2-1) \tag{151}$$

Rather, a physical state can only be annihilated by half of the Virasoro generators,

$$L_m\left|\phi\right\rangle \;=\; 0 \qquad \text{for} \qquad m>0 \tag{152}$$

Together with the mass-shell condition

$$\boxed{L_{m>0}\left|\phi\right\rangle \;=\; 0 \qquad \text{and} \qquad \big(L_0-a\big)\left|\phi\right\rangle \;=\; 0} \tag{153}$$

this characterizes a physical state $|\phi\rangle$. This is sufficient to give vanishing matrix elements of $L_n - a\delta_{n,0}$ between physical states, for all n.

$$\alpha_{-n} = (\alpha_n)^* \;\; \text{and} \;\; \tilde{\alpha}_{-n} = (\tilde{\alpha}_n)^* \qquad \Longrightarrow \qquad L_{-m} = L_m^\dagger \tag{154}$$

This ensures that negative-mode Virasoro operators annihilate physical states on their left

$$\left\langle\phi\right| L_m \;=\; 0 \qquad \text{for} \qquad m<0 \tag{155}$$

Mass operator

For the open string

$$\alpha' M^2 = \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n - a = N - a \qquad \Longrightarrow \qquad M^2 = \frac{N-a}{\alpha'} \tag{156}$$

where

$$N = \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{+\infty} n\, a_n^\dagger \cdot a_n \tag{157}$$

is called the **number operator**, since it has integer eigenvalues.

For the closed string,

$$\frac{1}{4}\alpha' M^2 = \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n - a = \sum_{n=1}^{+\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n - a = N - a = \tilde{N} - a \quad \Longrightarrow \quad M^2 = \frac{N-a}{4\alpha'} = \frac{\tilde{N}-a}{4\alpha'} \tag{158}$$

The normal-ordering constant a cancels out of the difference

$$\big(L_0 - a\big)\,|\phi\rangle = \big(\tilde{L}_0 - a\big)\,|\phi\rangle = 0 \qquad \Longrightarrow \qquad \big(L_0 - \tilde{L}_0\big)\,|\phi\rangle = 0 \tag{159}$$

which implies $N = \tilde{N}$. This is the so-called **level-matching condition** of the bosonic string.
It is the only constraint that relates the left- and right-moving modes.

There are no normal-ordering ambiguities in the Lorentz generators

$$\textbf{Open string:} \quad J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}\Big(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\Big) \tag{160}$$

and therefore they can be interpreted as quantum operators without any quantum corrections.

Using the canonical commutation relations Eq. (109) and Eq. (110), we may verify the Poincaré algebra

$$[p^\mu, p^\nu] = 0 \tag{161}$$

$$[p^\mu, J^{\nu\rho}] = -i\eta^{\mu\nu}p^\rho + i\eta^{\mu\rho}p^\nu \tag{162}$$

$$[J^{\mu\nu}, J^{\sigma\rho}] = i\eta^{\mu\sigma}J^{\nu\rho} + i\eta^{\nu\rho}J^{\mu\sigma} - i\eta^{\mu\rho}J^{\nu\sigma} - i\eta^{\nu\sigma}J^{\nu\rho} \tag{163}$$

So physical state conditions are variant under Lorentz transformations, and physical states are guaranteed to from Lorentz multiplets.

Next we verify the following formula for the open string:

$$\big[L_m, J^{\mu\nu}\big] = 0 \tag{164}$$

Recall following formulae:

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\alpha_{m-n}\cdot\alpha_n \tag{165}$$

First of all, we calculate the first two terms $l^{\mu\nu}$ in the right hand of Eq. (160)

$$
\begin{aligned}
\left[L_m,\, l^{\mu\nu}\right] &= \left[L_m,\, x^\mu p^\nu - x^\nu p^\mu\right] \\
&= \frac{1}{2}\sum_{n=-\infty}^{+\infty} \eta_{\rho\lambda}\left[\alpha^\rho_{m-n}\alpha^\lambda_n,\, x^\mu p^\nu - x^\nu p^\mu\right] \\
&= \frac{1}{2}\eta_{\rho\lambda}\left[\alpha^\rho_m\alpha^\lambda_0 + \alpha^\rho_0\alpha^\lambda_m,\, x^\mu p^\nu - x^\nu p^\mu\right] \\
&= \frac{1}{2}\eta_{\rho\lambda}\left[\alpha^\rho_m p^\lambda + p^\rho\alpha^\lambda_m,\, x^\mu \alpha^\nu_0 - x^\nu \alpha^\mu_0\right] \\
&= -i\left(\alpha^\mu_m\alpha^\nu_0 \,-\, \alpha^\nu_m\alpha^\mu_0\right)
\end{aligned} \tag{166}
$$

Next we calculate the second two terms (except a factor '$-i$')

$$\begin{aligned}
&\frac{1}{2}\sum_{n=-\infty}^{+\infty}\sum_{k=1}^{\infty}\frac{1}{k}\eta_{\rho\lambda}\left[\alpha_{m-n}^{\rho}\alpha_{n}^{\lambda},\ \alpha_{-k}^{\mu}\alpha_{k}^{\nu}-\alpha_{-k}^{\nu}\alpha_{k}^{\mu}\right] \\
=\ & \sum_{k=1}^{\infty}\left(\alpha_{m-k}^{\nu}\alpha_{k}^{\mu}\ -\ \alpha_{m-k}^{\mu}\alpha_{k}^{\nu}\ +\ \alpha_{-k}^{\mu}\alpha_{m+k}^{\nu}\ -\ \alpha_{-k}^{\nu}\alpha_{m+k}^{\mu}\right) \\
=\ & -\sum_{k=1}^{\infty}\left(\alpha_{m-k}^{\mu}\alpha_{k}^{\nu}\ -\ \alpha_{m-k}^{\nu}\alpha_{k}^{\mu}\right)\ -\ \sum_{k=-\infty}^{-1}\left(\alpha_{k}^{\nu}\alpha_{m-k}^{\mu}\ -\ \alpha_{k}^{\mu}\alpha_{m-k}^{\nu}\right) \\
=\ & -\sum_{k=1}^{\infty}\left(\alpha_{m-k}^{\mu}\alpha_{k}^{\nu}\ -\ \alpha_{m-k}^{\nu}\alpha_{k}^{\mu}\right)\ -\ \sum_{k=1}^{\infty}\left(\alpha_{m-k}^{\mu}\alpha_{k}^{\nu}+\left[\alpha_{k}^{\nu},\ \alpha_{m-k}^{\mu}\right]\ -\ \alpha_{m-k}^{\nu}\alpha_{k}^{\mu}-\left[\alpha_{k}^{\mu},\ \alpha_{m-k}^{\nu}\right]\right) \\
=\ & -\sum_{k=-\infty}^{\infty}\left(\alpha_{m-k}^{\mu}\alpha_{k}^{\nu}\ -\ \alpha_{m-k}^{\nu}\alpha_{k}^{\mu}\right)\ -\ \left(\alpha_{m}^{\mu}\alpha_{0}^{\nu}\ -\ \alpha_{m}^{\nu}\alpha_{0}^{\mu}\right) \\
=\ & -\ \left(\alpha_{m}^{\mu}\alpha_{0}^{\nu}\ -\ \alpha_{m}^{\nu}\alpha_{0}^{\mu}\right)
\end{aligned} \tag{167}$$

Therefore one obtains

$$\left[L_{m},\ J^{\mu\nu}\right]\ =\ 0 \tag{168}$$

This implies that the physical-state condition is invariant under Lorentz transformations. Therefore, physical states must appear in complete Lorentz multiplets. This follows from the fact that, the formalism being discussed here is manifestly Lorentz covariant.

No-ghost theorem

Next we make a preliminary investigation of the conditions that ensure that there are no negative-norm physical states. The result will be that there are negative-norm states for certain regions of the parameters a and the spacetime dimension D and not for other.

If one varies a and D to cross from a region where the physical Hilbert space is positive semi-definite to a region where it has negative-norm state, extra physical states of zero norm are always present at the boundaries between the two regions.

Let us consider a open string:

$$M^2 = -k^2 = \frac{N-a}{\alpha'} \tag{169}$$

First excited state, $N = 1$:

$$|\epsilon, k\rangle = \epsilon \cdot \alpha_{-1} |0; k\rangle \qquad\qquad k^2 \sim a - 1 \tag{170}$$

Using the physical condition $L_1 |\epsilon; k\rangle = 0$:

$$\begin{aligned} 0 = L_1 |\epsilon, k\rangle &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{1-n} \cdot \alpha_n \, \epsilon \cdot \alpha_{-1} |0; k\rangle \\ &= \alpha_0 \cdot \alpha_1 \, \epsilon \cdot \alpha_{-1} |0; k\rangle \\ &= (\alpha_0 \cdot \epsilon) |0; k\rangle \end{aligned} \tag{171}$$

$\Longrightarrow$

$$\epsilon \cdot k = 0 \tag{172}$$

Norm

$$\langle \epsilon, k | \epsilon, k \rangle = \epsilon \cdot \epsilon \tag{173}$$

Analysis:

- $a < 1$, $k^2 < 0$ timelike $\Longrightarrow$ ϵ^μ spacelike $\Longrightarrow$ $\texttt{Norm} = \epsilon \cdot \epsilon > 0$
- $a = 1$, $k^2 = 0$ null $\Longrightarrow$ $\epsilon = \epsilon_l + \epsilon_t$, ϵ_l null, ϵ_t spacelike $\Longrightarrow$ $\texttt{Norm} = \epsilon \cdot \epsilon = \epsilon_t \cdot \epsilon_t \geqslant 0$?
- $a > 1$, $k^2 > 0$ spacelike $\Longrightarrow$ ϵ^μ timelike $\Longrightarrow$ $\texttt{Norm} = \epsilon \cdot \epsilon < 0$

Thus we obtain the first condition for the absence of ghosts:

$$a \leq 1 \tag{174}$$

In the boundary case, $a = 1$, the vector particle is massless and the scalar particle is tachyon. The L_1 subsidiary condition corresponds to the covariant gauge condition $\partial_\mu A^\mu = 0$ of electrodynamics. This condition leaves $D - 2$ positive states with transverse polarization and one longitudinal state $\epsilon^\mu = k^\mu$ of zero norm.

Spurious states

To analyze the spectrum, we define a so-called spurious state. A state $|\psi\rangle$ is called **spurious** if it satisfies the mass-shell condition and is orthogonal to all physical states

$$(L_0 - a)|\psi\rangle = 0 \qquad \text{and} \qquad \langle\phi|\psi\rangle = 0 \tag{175}$$

where $|\phi\rangle = |\texttt{physics}\rangle$ represents any physical state in the theory. An example of a spurious state is

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle \qquad \text{with} \qquad (L_0 - a + n)|\chi_n\rangle = 0 \tag{176}$$

First let us verify this state satisfies the mass-shell condition

$$\begin{aligned}
(L_0 - a)|\psi\rangle &= (L_0 - a)\sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle \\
&= \sum_{n=1}^{\infty}(L_0 L_{-n} - aL_{-n})|\chi_n\rangle \qquad [L_0, L_{-n}] = nL_{-n} \\
&= \sum_{n=1}^{\infty}(L_{-n}L_0 + nL_{-n} - aL_{-n})|\chi_n\rangle \\
&= \sum_{n=1}^{\infty} L_{-n}(L_0 + n - a)|\chi_n\rangle \\
&= 0
\end{aligned} \tag{177}$$

Then we show it is orthogonal to all physical states

$$\langle\phi|\psi\rangle \;=\; \sum_{n=1}^{\infty}\langle\phi|L_{-n}\,|\chi_n\rangle \;=\; 0 \tag{178}$$

In fact, any such state can be recast in the form

$$|\psi\rangle \;=\; L_{-1}\,|\chi_1\rangle \,+\, L_{-2}\,|\chi_2\rangle \tag{179}$$

as a consequence of the Virasoro algebra (e.g. $L_{-3} = [L_{-1}, L_{-2}]$). Moreover, any spurious state can be put in this form.

If a state $|\psi\rangle$ is spurious and physical, then it is orthogonal to all physical states including itself

$$\langle\psi|\psi\rangle \;=\; \sum_{n=1}^{\infty}\langle\psi|L_{-n}\,|\chi_n\rangle \;=\; \sum_{n=1}^{\infty}\langle\chi_n|L_n\,|\psi\rangle \;=\; 0 \tag{180}$$

Determination of a

When the constant a is suitably chosen, a class of zero-norm spurious states has the form

$$|\psi\rangle \;=\; L_{-1}\,|\chi_1\rangle \tag{181}$$

with

$$(L_0 - a + 1)\,|\chi_1\rangle = 0 \qquad \text{and} \qquad L_n\,|\chi_1\rangle = 0, \quad n > 0 \tag{182}$$

Demanding that $|\psi\rangle$ is physical implies

$$L_n |\psi\rangle = (L_0 - a) |\psi\rangle = 0 \qquad \text{for} \qquad n = 1, 2, \ldots \tag{183}$$

Note that the Virasoro algebra

$$[L_1,\, L_{-1}] = 2L_0 \qquad \Longrightarrow \qquad L_1 L_{-1} = 2L_0 + L_{-1}L_1 \tag{184}$$

We have

$$0 = L_1 |\psi\rangle = L_1 L_{-1} |\chi_1\rangle = \big(2L_0 + L_{-1}L_1\big) |\chi_1\rangle = 2L_0 |\chi_1\rangle = 2(a-1) |\chi_1\rangle \tag{185}$$

$\Longrightarrow$

$$a \;=\; 1 \tag{186}$$

Thus $a = 1$ is part of the specification of the boundary between positive-norm and negative-norm physical states. For details, see GSW [1], section 2.2.

Determination of the space-time dimension D

The number of zero-norm spurious states increases dramatically if, in addition to $a = 1$, the space-time dimension is chosen appropriately. To see this, let us construct zero-norm spurious states of the form

$$|\psi\rangle \;=\; \big(L_{-2} + \gamma L_{-1}^2\big) |\tilde{\chi}\rangle \tag{187}$$

This has zero norm for a certain γ, which is determined below. Here ψ is spurious if $|\tilde{\chi}\rangle$ is a state that satisfies

$$(L_0+1)\,|\tilde{\chi}\rangle = L_m\,|\tilde{\chi}\rangle = 0 \qquad \text{and} \qquad n = 1, 2, \dots \tag{188}$$

Now we verify this statement

$$\begin{aligned}
(L_0-a)\,|\psi\rangle \;=\; (L_0-1)\,|\psi\rangle &= (L_0-1)\big(L_{-2}+\gamma L_{-1}^2\big)\,|\tilde{\chi}\rangle \\
&= \Big(L_0L_{-2}-L_{-2}+\gamma\big(L_0L_{-1}^2-L_{-1}^2\big)\Big)\,|\tilde{\chi}\rangle \\
&= \Big(L_{-2}L_0+2L_{-2}-L_{-2}+\gamma\big(L_0L_{-1}^2-L_{-1}^2\big)\Big)\,|\tilde{\chi}\rangle \\
&= \gamma\big(L_0L_{-1}^2-L_{-1}^2\big)\,|\tilde{\chi}\rangle \\
&= \gamma\big(L_{-1}^2L_0+2L_{-1}^2-L_{-1}^2\big)\,|\tilde{\chi}\rangle \;=\; \gamma\,L_{-1}^2\big(L_0+1\big)\,|\tilde{\chi}\rangle \\
&= 0
\end{aligned} \tag{189}$$

We use Varasoro algebra in the 5th line

$$[L_0,\,L_{-1}^2] \;=\; L_{-1}[L_0,\,L_{-1}]+[L_0,\,L_{-1}]L_{-1} \;=\; L_{-1}^2+L_{-1}^2 \;=\; 2L_{-1}^2 \tag{190}$$

Now impose the condition that $|\psi\rangle$ is a physical state, i.e. $L_n\,|\psi\rangle=0,\;n\in\mathbb{N}^+$. In fact, we only need to consider the case of $n=1$ and $n=2$, that is

$$L_1\,|\psi\rangle = 0 \qquad \text{and} \qquad L_2\,|\psi\rangle = 0, \tag{191}$$

since the rest of the constraints $L_n |\psi\rangle = 0$ for $n > 3$ are then also satisfied as a consequence of the Virasoro algebra. For example, $L_3 = [L_2, L_1]$.

Let us first evaluate the condition $L_1 |\psi\rangle = 0$. Notice that

$$[L_1,\ L_{-2}] = 3L_{-1} \tag{192}$$

$$[L_1,\ L_{-1}^2] = L_{-1}[L_1,\ L_{-1}] + [L_1,\ L_{-1}]L_{-1} = 2L_{-1}L_0 + 2L_0L_{-1} = 4L_0L_{-1} - 2L_{-1} \tag{193}$$

Thus we have

$$\begin{aligned} 0 \;=\; L_1 |\psi\rangle &= L_1\big(L_{-2} + \gamma L_{-1}^2\big) |\tilde{\chi}\rangle \\ &= \Big(L_{-2}L_1 + 3L_{-1} + \gamma\big(L_{-1}^2 L_1 + 4L_0L_{-1} - 2L_{-1}\big)\Big) |\tilde{\chi}\rangle \\ &= \Big(3L_{-1} + \gamma\big(4L_0L_{-1} - 2L_{-1}\big)\Big) |\tilde{\chi}\rangle \\ &= \Big((3-2\gamma)L_{-1} \;+\; 4\gamma L_0L_{-1}\Big) |\tilde{\chi}\rangle \\ &= (3-2\gamma)L_{-1} |\tilde{\chi}\rangle \;+\; 4\gamma\big(L_{-1}L_0 + L_{-1}\big) |\tilde{\chi}\rangle \\ &= (3-2\gamma)L_{-1} |\tilde{\chi}\rangle \end{aligned} \tag{194}$$

$\Longrightarrow$

$$\gamma \;=\; \frac{3}{2} \tag{195}$$

Let us next consider the second condition $L_2 \left|\psi\right\rangle = 0$. Similarly, we first calculate

$$[L_2,\ L_{-2}] = 4L_0 + \frac{c}{12} \times 2 \times (4-1) = 4L_0 + \frac{c}{2} \tag{196}$$

$$[L_2,\ L_{-1}^2] = L_{-1}[L_2,\ L_{-1}] + [L_2,\ L_{-1}]L_{-1} = 3\big(L_{-1}L_1 + L_1L_{-1}\big) = 6\big(L_0 + L_{-1}L_1\big) \tag{197}$$

Therefore

$$\begin{aligned} 0 \ = \ L_2 \left|\psi\right\rangle \ &= \ L_2\big(L_{-2} + \tfrac{3}{2}L_{-1}^2\big) \left|\tilde{\chi}\right\rangle \\ &= \ \left(4L_0 + \frac{c}{2} + 9\big(L_0 + L_{-1}L_1\big)\right) \left|\tilde{\chi}\right\rangle \\ &= \ \left(13L_0 + \frac{c}{2} + 9L_{-1}L_1\right) \left|\tilde{\chi}\right\rangle \\ &= \ \left(-13 + \frac{c}{2}\right) \left|\tilde{\chi}\right\rangle \end{aligned} \tag{198}$$

$\Longrightarrow$

$$D \ = \ c \ = \ 26 \tag{199}$$

Thus the space-time dimension $D = 26$ gives additional zero-norm spurious states.

Next we will explain that physical states of negative norm can be constructed for $D > 26$. Let us consider a state of the form

$$|\phi\rangle \; = \; \Big(A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_{-2} + C\big(\alpha_0\cdot\alpha_{-1}\big)^2\Big)\,|0;k\rangle \tag{200}$$

- $a = 1$, we have

$$0 \; = \; \big(L_0 - 1\big)\,|\phi\rangle \; = \; \left(\frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n\right)|\phi\rangle \tag{201}$$

In fact, we just calculate one term of the right hand side of the Eq. (200).

$$\begin{aligned} 0 \; &= \; \big(L_0 - 1\big)\alpha_0\cdot\alpha_{-2}\,|0;k\rangle \\ &= \; \Big(\tfrac{1}{2}\alpha_0^2 + \alpha_{-2}\cdot\alpha_2 \; - 1\Big)\alpha_0\cdot\alpha_{-2}\,|0;k\rangle \\ &= \; \Big(\tfrac{1}{2}\alpha_0^2 + 2 - 1\Big)\alpha_0\cdot\alpha_{-2}\,|0;k\rangle \end{aligned} \tag{202}$$

So we have $\alpha_0^2 = -2$.

- $L_1 |\phi\rangle = 0$

$$\sum_{n=-\infty}^{\infty} \alpha_{1-n} \cdot \alpha_n \, \alpha_{-1} \cdot \alpha_{-1} |0;k\rangle \;=\; 2\alpha_0 \cdot \alpha_1 \, \alpha_{-1} \cdot \alpha_{-1} |0;k\rangle \;=\; 4\alpha_0 \cdot \alpha_{-1} |0;k\rangle \tag{203}$$

$$\sum_{n=-\infty}^{\infty} \alpha_{1-n} \cdot \alpha_n \, \alpha_0 \cdot \alpha_{-2} |0;k\rangle \;=\; 2\alpha_{-1} \cdot \alpha_2 \, \alpha_0 \cdot \alpha_{-2} |0;k\rangle \;=\; 4\alpha_0 \cdot \alpha_{-1} |0;k\rangle \tag{204}$$

$$\sum_{n=-\infty}^{\infty} \alpha_{1-n} \cdot \alpha_n \, (\alpha_0 \cdot \alpha_{-1})^2 |0;k\rangle \;=\; 2\alpha_0 \cdot \alpha_1 \, (\alpha_0 \cdot \alpha_{-1})^2 |0;k\rangle \;=\; -8\alpha_0 \cdot \alpha_{-1} |0;k\rangle \tag{205}$$

Using above formula, we have

$$0 \;=\; L_1 |\phi\rangle \;=\; 2\big(A + B - 2C\big)\alpha_0 \cdot \alpha_{-1} |0;k\rangle \tag{206}$$

$\Longrightarrow$

$$A + B - 2C \;=\; 0 \tag{207}$$

- $L_2 \left|\phi\right\rangle = 0$

$$\sum_{n=-\infty}^{\infty} \alpha_{2-n} \cdot \alpha_n \, \alpha_{-1} \cdot \alpha_{-1} \left|0;k\right\rangle \;=\; \alpha_1 \cdot \alpha_1 \, \alpha_{-1} \cdot \alpha_{-1} \left|0;k\right\rangle \;=\; 2D \left|0;k\right\rangle \tag{208}$$

$$\sum_{n=-\infty}^{\infty} \alpha_{2-n} \cdot \alpha_n \, \alpha_0 \cdot \alpha_{-2} \left|0;k\right\rangle \;=\; 2\alpha_0 \cdot \alpha_2 \, \alpha_0 \cdot \alpha_{-2} \left|0;k\right\rangle \;=\; -8 \left|0;k\right\rangle \tag{209}$$

$$\sum_{n=-\infty}^{\infty} \alpha_{2-n} \cdot \alpha_n \left(\alpha_0 \cdot \alpha_{-1}\right)^2 \left|0;k\right\rangle \;=\; 2\alpha_1 \cdot \alpha_1 \left(\alpha_0 \cdot \alpha_{-1}\right)^2 \left|0;k\right\rangle \;=\; -8 \left|0;k\right\rangle \tag{210}$$

Therefore we have

$$0 \;=\; L_2 \left|\phi\right\rangle \;=\; \big(DA - 4B - 2C\big) \left|0;k\right\rangle \tag{211}$$

$\Longrightarrow$

$$DA - 4B - 2C \;=\; 0 \tag{212}$$

Thus

$$B \;=\; \frac{D-1}{5} A, \qquad C \;=\; \frac{D+4}{10} A \tag{213}$$

In this case, the norm is

$$\left\langle\phi\middle|\phi\right\rangle \;=\; \frac{2A^2}{25}(D-1)(26-D) \tag{214}$$

so that we find ghosts in the physical spectrum for $D > 26$.

The zero-norm spurious states are unphysical. The fact that they are spurious ensures that they decouple from all physical processes. In fact, all negative-norm states decouple, and all physical states have positive norm. Thus, the complete physical spectrum is free of negative-norm states when the two conditions $a = 1$ and $D = 26$ are satisfied.

In fact, the physical spectrum is also ghost in a region: $a \leq 1$ and $D < 26$.

No Ghost Theorem

- For $D > 26$ **or** $a > 1$, the physical space contains negative norm states;
- For $D = 26$ **and** $a < 1$, the physical space contains negative norm states;
- For $D = 26$ **and** $a = 1$, the physical space contains *only* non-negative norm states, in particular, two towers of null states and $\mathcal{F}^{\oplus} \equiv \mathcal{F}^{\text{phys}}/\mathcal{F}^0$ has positive definite norm;

$$\mathcal{F}^0 = \left\{ \left(L_{-2} + \tfrac{3}{2}L_{-1}^2\right)|\tilde{\chi}\rangle \oplus L_{-1}|\chi_1\rangle \right\} \tag{215}$$

- For $D < 26$ **and** $a \leq 1$, the physical space $\mathcal{F}^{\text{phys}}$ contains only positive norm states for $a < 1$ (contains one tower of null states $L_{-1}|\chi_1\rangle$ for $a = 1$).

The $a = 1$, $D = 26$ bosonic string theory is called critical, and one says that the critical dimension is 26. In the case of $a \leqslant 1$ and $D \leqslant 25$, the theory is called noncritical.

3 Light-cone gauge quantization

The light-cone coordinates for space-time

First of all, introduce light-cone coordinates for space-time

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 + X^{D-1}\right) \tag{216}$$

Then the D space-time coordinates X^μ consist of the null coordinates $X^\pm$ and the $D-2$ transverse coordinates X^i. In this notation, the inner product of two arbitrary vectors takes the form

$$\begin{aligned} V\cdot W = V^\mu W_\mu &= -V^0W^0 + V^{D-1}W^{D-1} + \sum_{i=1}^{D-2} V^iW^i \\ &= -\frac{1}{2}\left(V^+ + V^-\right)\left(W^+ + W^-\right) + \frac{1}{2}\left(V^+ - V^-\right)\left(W^+ - W^-\right) + \sum_{i=1}^{D-2} V^iW^i \\ &= -V^+W^- - V^-W^+ + \sum_{i=1}^{D-2} V^iW^i \end{aligned} \tag{217}$$

Indices are raised and lowered by the rules

$$V^+ = -V_- \qquad V^- = -V_+ \qquad V^i = V_i \tag{218}$$

When we use light-cone coordinates, Lorentz invariance is no longer manifest.

The gauge fixing revisited

The **conformal gauge**:

$$h_{ab} \xrightarrow{\texttt{Diff}} (h_{ab}) = e^{\rho}(\eta_{ab}) \tag{219}$$

Using Weyl symmetry, we can further reduce the metric h_{ab} to a more simple form

$$h_{ab} = \eta_{ab} \tag{220}$$

Thus, the string action takes a simple form

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}\,h^{ab}\,\partial_a X^{\mu}\partial_b X_{\mu} = -\frac{T}{2}\int d^2\sigma\,\eta^{ab}\,\partial_a X^{\mu}\partial_b X_{\mu} \tag{221}$$

The equation of motion for X^{μ} becomes

$$\eta^{ab}\partial_a\partial_b X^{\mu} = 0 \qquad \text{or} \qquad \left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right)X^{\mu} = -\Box X^{\mu} = 0 \tag{222}$$

The equation of motion for h_{ab} becomes the constraints:

$$\begin{aligned} 0 = T_{ab} &= \partial_a X^{\mu}\partial_b X_{\mu} - \frac{1}{2}h_{ab}\,h^{cd}\,\partial_c X^{\mu}\partial_d X_{\mu} \\ &= \partial_a X^{\mu}\partial_b X_{\mu} - \frac{1}{2}\eta_{ab}\,\eta^{cd}\,\partial_c X^{\mu}\partial_d X_{\mu} \end{aligned} \tag{223}$$

$\Longrightarrow$

$$\textbf{Virasoro constraints}: \qquad \left(\dot{X} \pm X'\right)^2 = 0 \tag{224}$$

However, as discussed earlier, the bosonic string has residual diffeomorphism symmetries which consist of all the conformal equivalence class.

$$\sigma^{\pm} \longrightarrow \xi^{\pm}(\sigma^{\pm}) \tag{225}$$

Accordingly, the metric tensor becomes

$$\tilde{h}_{\alpha\beta} = \frac{\partial\sigma^a}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^b}{\partial\tilde{\sigma}^\beta}(e^\rho\eta_{ab}) \tag{226}$$

For convenience, we work on the light-cone coordinates,

$$\begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{227}$$

$\Longrightarrow$

$$\begin{aligned} \tilde{h}_{\alpha\beta} &= \frac{\partial\sigma^a}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^b}{\partial\tilde{\sigma}^\beta}(e^\rho\eta_{ab}) \\ &= \frac{1}{2}e^\rho\frac{\partial\sigma^+}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^-}{\partial\tilde{\sigma}^\beta} + \frac{1}{2}e^\rho\frac{\partial\sigma^-}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^+}{\partial\tilde{\sigma}^\beta} \end{aligned} \tag{228}$$

$\Longrightarrow$

$$\tilde{h}_{++} = \tilde{h}_{--} = 0, \qquad \tilde{h}_{+-} = \tilde{h}_{-+} = \frac{1}{2}e^\rho\frac{\partial\sigma^+}{\partial\tilde{\sigma}^+}\frac{\partial\sigma^-}{\partial\tilde{\sigma}^-} = \frac{1}{2}e^\Lambda \tag{229}$$

Therefore, the residual symmetry transformation $\sigma^{\pm} \to \xi^{\pm}(\sigma^{\pm})$ respects the gauge choice $h_{ab} = e^\rho\eta_{ab}$.

In terms of $\sigma^{\pm}$ the residual symmetry corresponds to the reparametrizations

$$\sigma^{\pm} \longrightarrow \xi^{\pm}(\sigma^{\pm}) \tag{230}$$

These transformations correspond to

$$\tau \longrightarrow \tilde{\tau} = \frac{1}{2}\Big(\xi^{+}(\sigma^{+}) + \xi^{-}(\sigma^{-})\Big) \tag{231}$$

$$\sigma \longrightarrow \tilde{\sigma} = \frac{1}{2}\Big(\xi^{+}(\sigma^{+}) - \xi^{-}(\sigma^{-})\Big) \tag{232}$$

It is easy to verify

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right)\tilde{\tau} = -\Box\tilde{\tau} = 0 \tag{233}$$

On the other hand, in conformal gauge the scalar field X^{μ} also satisfy the two-dimensional wave equation $\Box X^{\mu} = 0$ and the general solutions of the X^{μ} are

$$X^{\mu}(\tau,\sigma) = f^{\mu}(\sigma^{+}) + g^{\mu}(\sigma^{-}) \tag{234}$$

So a natural choice of the solution of $\tilde{\tau}$ is

$$\sigma^{\pm} \xRightarrow{\texttt{Diff}} \xi^{\pm}(\sigma^{\pm}):\ \tilde{\tau} = \frac{1}{2}\Big(\xi^{+}(\sigma^{+}) + \xi^{-}(\sigma^{-})\Big) \sim f^{\mu}(\sigma^{+}) + g^{\mu}(\sigma^{-}) \tag{235}$$

$\Longrightarrow$

$$\tilde{\tau} \sim \text{one of the } X^{\mu} \tag{236}$$

The light-cone gauge corresponds to the choice

$$\tilde{\tau} \;=\; \frac{X^+}{2\alpha' p^+} + \text{constant} \qquad \Longrightarrow \qquad \boxed{X^+(\tilde{\tau},\tilde{\sigma}) \;=\; x^+ \,+\, 2\alpha' p^+\,\tilde{\tau}} \tag{237}$$

In the following the tildes are omitted from the parameters $\tilde{\tau}$ and $\tilde{\sigma}$, namely replace

$$(\tilde{\tau},\,\tilde{\sigma}) \to (\tau,\,\sigma) \qquad \Longrightarrow \qquad X^+(\tau,\sigma) \;=\; x^+ \,+\, 2\alpha' p^+\tau \tag{238}$$

This corresponds to setting

$$\alpha^+ \;=\; 0 \quad \text{for} \quad n \neq 0 \tag{239}$$

The light-cone gauge has eliminated the oscillator modes of X^+. It is possible to determine the oscillator modes of X^-. Using Virasoro constraints

$$\begin{aligned} 0 \;=\; \left(\dot{X} \pm X'\right)^2 \;&=\; -\left(\dot{X}^+ \pm X^{+\prime}\right)\left(\dot{X}^- \pm X^{-\prime}\right) - \left(\dot{X}^- \pm X^{-\prime}\right)\left(\dot{X}^+ \pm X^{+\prime}\right) + \sum_i \left(\dot{X}^i \pm X^{i\prime}\right)^2 \\ &=\; -4\alpha' p^+\left(\dot{X}^- \pm X^{-\prime}\right) \,+\, \sum_i \left(\dot{X}^i \pm X^{i\prime}\right)^2 \end{aligned} \tag{240}$$

$\Longrightarrow$

$$\left(\dot{X}^- \pm X^{-\prime}\right) \;=\; \frac{1}{4\alpha' p^+}\sum_i \left(\dot{X}^i \pm X^{i\prime}\right)^2 \tag{241}$$

Note that the following formula for the open string

$$2\partial_\pm X^\mu \;=\; \dot{X}^\mu \pm X'^\mu \;=\; \sqrt{2\alpha'} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu \, e^{-in(\tau\pm\sigma)} \tag{242}$$

we have

$$\left(\dot{X}^- \pm X^{-\prime}\right) \;=\; \frac{1}{4\alpha' p^+} \sum_i \left(\dot{X}^i \pm X^{i\prime}\right)^2 \quad \Longrightarrow \quad \partial_\pm X^- \;=\; \frac{1}{2\alpha' p^+} \sum_i \left(\partial_\pm X^i\right)^2 \tag{243}$$

This pair of equations can be used to solve for X^- in terms of X^i. In terms of the mode expansion for X^-, which for an open string is

$$X^- \;=\; x^- \,+\, 2\alpha' p^- \tau \,+\, i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^- \, e^{-in\tau} \cos n\sigma \tag{244}$$

So we can easily obtain the left hand side of the Eq. (243):

$$\partial_+ X^- \;=\; \sqrt{\alpha'/2} \sum_{n=-\infty}^{\infty} \alpha_n^- \, e^{-in(\tau+\sigma)} \qquad \alpha_0^- \;=\; \sqrt{2\alpha'}\, p^- \tag{245}$$

Thus the right hand side of the Eq. (243) is

$$\begin{aligned} \frac{1}{2\alpha' p^+} \sum_i \left(\partial_+ X^i\right)^2 &= \frac{1}{4p^+} \sum_i \sum_k \sum_m \alpha_k^i \alpha_m^i \, e^{-i(k+m)(\tau+\sigma)} = \frac{1}{4p^+} \sum_i \sum_n \sum_m \alpha_{n-m}^i \alpha_m^i \, e^{-in(\tau+\sigma)} \\ &= \frac{1}{4p^+} \sum_n \sum_i \sum_m \alpha_{n-m}^i \alpha_m^i \, e^{-in(\tau+\sigma)} \end{aligned} \tag{246}$$

Therefore we obtain the explicit solution:

$$
\begin{aligned}
\alpha_n^- &= \frac{1}{2\sqrt{2\alpha'}\,p^+}\sum_{i=1}^{D-2}\sum_m \alpha_{n-m}^i\alpha_m^i = \frac{1}{2\alpha_0^+}\sum_{i=1}^{D-2}\sum_m \alpha_{n-m}^i\alpha_m^i \\
&= \frac{1}{\sqrt{2\alpha'}p^+}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_m :\alpha_{n-m}^i\alpha_m^i: -a\,\delta_{n,0}\right), \qquad (247)
\end{aligned}
$$

For closed string theory the formulas are almost the same with both left and right movers:

$$
X^+(\tau,\sigma) = x^+ + \alpha'\,p^+\,\tau, \qquad (248)
$$

$$
X^-(\tau,\sigma) = x^- + \alpha'\,p^-\,\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^-\,\mathrm{e}^{-in(\tau+\sigma)} + \tilde{\alpha}_n^-\,\mathrm{e}^{-in(\tau-\sigma)}\right] \qquad (249)
$$

where

$$
\alpha_n^- = \frac{1}{\sqrt{\alpha'/2}\,p^+}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_m :\alpha_{n-m}^i\alpha_m^i: -a\,\delta_{n,0}\right), \qquad (250)
$$

$$
\tilde{\alpha}_n^- = \frac{1}{\sqrt{\alpha'/2}\,p^+}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_m :\tilde{\alpha}_{n-m}^i\tilde{\alpha}_m^i: -a\,\delta_{n,0}\right). \qquad (251)
$$

Therefore in light-cone gauge both X^+ and X^- can be eliminated, leaving only the transverse oscillators X^i.

Mass-shell condition

In the light-cone gauge the open-string mass-shell condition (156) is

$$M^2 = -p^2 = 2p^+p^- - \sum_i p_i^2 = \frac{N-a}{\alpha'} \tag{252}$$

where

$$N = \sum_{i=1}^{D-1} \sum_{n=1}^{+\infty} \alpha_{-n}^i \alpha_n^i \tag{253}$$

For the closed string (158),

$$M^2 = \frac{4(N-a)}{\alpha'} = \frac{4(\tilde{N}-a)}{\alpha'} \tag{254}$$

where

$$N = \sum_{i=1}^{D-1} \sum_{n=1}^{+\infty} \alpha_{-n}^i \alpha_n^i, \qquad \tilde{N} = \sum_{i=1}^{D-1} \sum_{n=1}^{+\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \tag{255}$$

We also need to consider the level-matching condition for the closed string

$$N = \tilde{N}. \tag{256}$$

Normal-ordering constant a

In the light-cone gauge all the excitations are generated by acting with the transverse modes α_n^i. First excited state ($N=1$): $\alpha_{-1}^i\,|0;k\rangle$, they form a $(D-2)$-component vector representation of the transverse rotation group $\mathrm{SO}(D-2)$.

As a general rule, Lorentz invariance implies that physical states form representations of $\mathrm{SO}(D-1)$ for massive states and $\mathrm{SO}(D-2)$ for massless states. As an example, we present the little group and single particle states of the Lorentz group in $D=4$ [23].

	Standard k^μ	Little group	Physical interpretation
$p^2=-M^2<0$, $p^0>0$	$(M,0,0,0)$	$\mathrm{SO}(3)$	massive positive-energy state
$p^2=-M^2<0$, $p^0<0$	$(-M,0,0,0)$	$\mathrm{SO}(3)$	massive negative-energy state
$p^2=0$, $p^0>0$	$(\kappa,0,0,\kappa)$	$\mathrm{ISO}(2)$	massless positive-energy state
$p^2=0$, $p^0<0$	$(-\kappa,0,0,\kappa)$	$\mathrm{ISO}(2)$	massless negative-energy state
$p^2=N^2>0$	$(0,0,0,N)$	$\mathrm{SO}(2,1)$	tachyon
$p^\mu=0$	$(0,0,0,0)$	$\mathrm{SO}(3,1)$	vacuum

Therefore, the bosonic string theory in the light-cone gauge can only be Lorentz invariant if the vector state $\alpha_{-1}^i\,|0;k\rangle$ is massless.

$$0 = M^2 = \frac{1-a}{\alpha'} \qquad \Longrightarrow \qquad a = 1 \tag{257}$$

Critical Dimension D

The next goal is to determine the spacetime dimension D. A heuristic approach is to compute the normal-ordering constant appearing in the definition of L_0 directly. This normal-ordering constant arises from the formula

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{+\infty}\alpha_{-n}^{i}\alpha_{n}^{i} &= \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{+\infty}\Big(:\alpha_{-n}^{i}\alpha_{n}^{i}: + \big[\alpha_{n}^{i},\, \alpha_{-n}^{i}\big]\Big) \\ &= \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{+\infty}:\alpha_{-n}^{i}\alpha_{n}^{i}: \;+\; \frac{1}{2}(D-2)\sum_{n=1}^{\infty}n \end{aligned} \tag{258}$$

The second sum on the right-hand side is divergent and needs to be regularized. This can be achieved using ζ-function regularization. Define the ζ-function as a sum

$$\zeta(s) = \sum_{n=1}^{\infty}n^{-s} \tag{259}$$

This converges for $\mathrm{Re}(s) > 1$ and has a pole at $s = 1$. It can be continued around the pole, and $\zeta(-1) = -\dfrac{1}{12}$. So this is the value we assign to the $\sum_{n=1}^{\infty}n = -\frac{1}{12}$, by analogy to dimensional regularization [3].

Here we give a simple illustration. Introduce an ultra-violet cut-off ϵ, $\epsilon \ll 1$

$$
\begin{aligned}
\sum_{n=1}^{\infty} n \;\longrightarrow\; \sum_{n=1}^{\infty} n\, e^{-\epsilon n} &= -\frac{d}{d\epsilon} \sum_{n=1}^{\infty} e^{-\epsilon n} = -\frac{d}{d\epsilon} \frac{1}{1 - e^{-\epsilon}} \\
&= \frac{e^{-\epsilon}}{\left(1 - e^{-\epsilon}\right)^2} \\
&= \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{260}
$$

Weyl invariance requires that we drop exactly the ϵ^{-2} term [3, 7, 8].

Thus we have

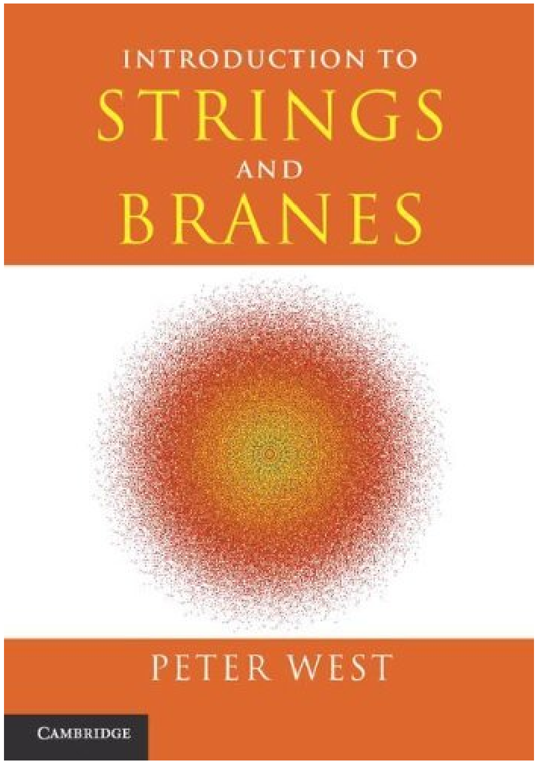
$$
\frac{1}{2}(D-2) \sum_{n=1}^{\infty} n = -\frac{D-2}{24} \tag{261}
$$

Using the earlier result that the normal-ordering constant $a = 1$, we have

$$
\frac{D-2}{24} = 1 \qquad \Longrightarrow \qquad D = 26 \tag{262}
$$

∗Lorentz symmetry

Another approach of the determination of a and D is to verify that the Lorentz generators satisfy the Lorentz algebra. We will find the algebra is closed only if $a = 1$ and $D = 26$. For details, see Peter West 2012 [20], Introduction to Strings and Branes, section 4.3, or Gleb Arutyunov 2008 [25], Lectures on String Theory, subsection 4.2.1.



The original calculation is done by GGRT: [21] P. Goddard, J. Goldstone, C. Rebbi, C. Thorn, *Quantum dynamics of a massless relativistic string*, NPB 56 (1973) 109.

I strongly recommend to you several nice lecture notes:

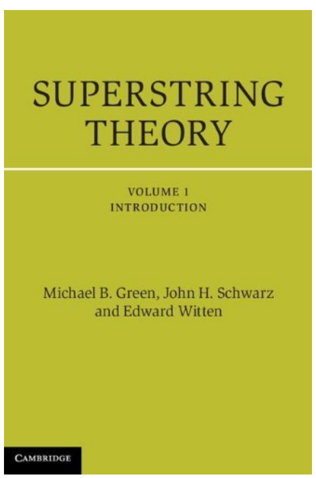
[7] Joseph Polchinski, ***Joe's Little Book of String***, 2010

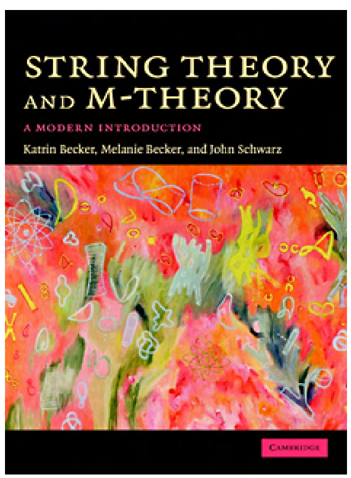
[8] David Tong, ***Lectures on String Theory***, 0908.0333

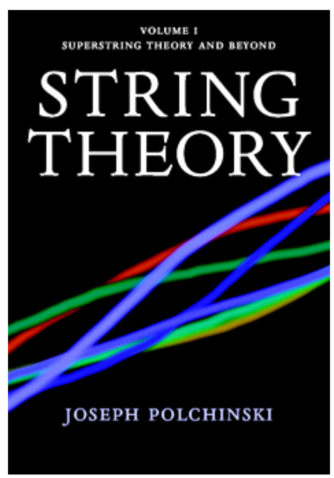
[9] Gerard 't Hooft, ***Introduction to String Theory***, 2004

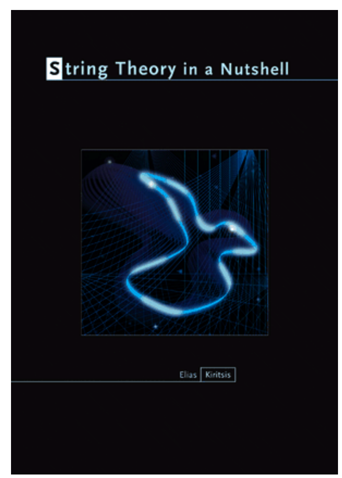
[25] Gleb Arutyunov, ***Lectures on String Theory***, Utrecht University, 2008

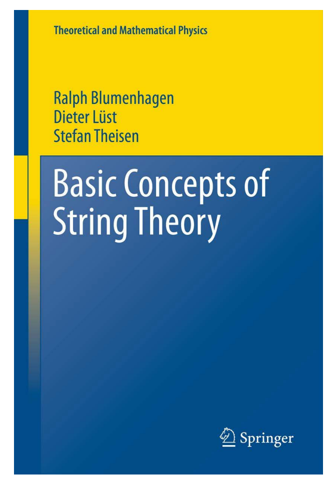
> When I wrote the Big Book of String, I had many goals. One was to make it very readable, so you would pick it up, be unable to put it down, and after staying up all night reading you would know string theory. But in spite of much effort, this didn't happen. The desire to be general and systematic pulled in the opposite direction. So these notes are intended to be the book I might have written, and I can leave many details to the Big Book.
>
> —Joseph Polchinski, Joe's Little Book of String











Analysis of the spectrum for the open string

Now we can determine the spectrum of the bosonic string. Recall the open-string mass-shell condition (252) in the light-cone gauge

$$M^2 = -p^2 = 2p^+p^- - \sum_i p_i^2 = \frac{N-a}{\alpha'}, \qquad N = \sum_{i=1}^{D-1}\sum_{n=1}^{+\infty} \alpha_{-n}^i \alpha_n^i \tag{263}$$

At the first few mass levels the physical states of the open string are as follows:

- $N = 0$, there is a state $|0;k\rangle$ which has $M^2 = -\frac{1}{\alpha'}$. This is a **tachyon** state.
- $N = 1$, the states are $\alpha_{-1}^i|0;k\rangle$ which is a vector boson. Lorentz invariance requires that it is massless, $M^2 = 0$. These states gives a vector representation of $\mathrm{SO}(24)$.
- $N = 2$, gives the first states with positive mass square. They are

$$\begin{array}{ccc} \alpha_{-2}^i\,|0;k\rangle & \alpha_{-1}^i\alpha_{-1}^j\,|0;k\rangle & \\ 24 & (24\times 25)/2 = 300 & \end{array} \qquad M^2 = \frac{1}{\alpha'}$$

  There are $(D-2) + \frac{(D-2)(D-1)}{2} = \frac{(D-2)(D+1)}{2} = 324$ different states which form a representation of $SO(25)$. $\frac{(D-2)(D+1)}{2} = \frac{(D-1)D}{2} - 1 = \frac{(D-1)(D-1+1)}{2} - 1$ is just the dimensionality of the symmetric traceless second-rank tensor representation of $\mathrm{SO}(25)$. So, in this sense, the spectrum consists of a single massive spin-two state at this mass level.

Representation of SO(N) group

For rank 0 we still have the 1-dimensional trivial representation, and for rank 1 we have the N-dimensional vector representation.

The rank 2 case we have two parts, **the symmetric traceless part** with dimension $N(N+1)/2-1$ and **the antisymmetric part** with dimension $N(N-1)/2$. For higher rank we need to use the Young tableaux.

- $N=3$, the possible states are

$$\alpha^i_{-3}\,|0;k\rangle \qquad \alpha^i_{-2}\alpha^j_{-1}\,|0;k\rangle \qquad \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\,|0;k\rangle \qquad M^2=\frac{2}{\alpha'}$$

  There are $24+576+2600=3200$ different states. Similarly, for $N=4$ there are totally $20150+5175+324+1=15650$ states, etc [1].

All of these states have a positive norm, since they are built entirely from the transverse modes, which describe a positive-definite Hilbert space. In the light-cone gauge the fact that the negative-norm states have decoupled is made manifest. All of the massive representations can be rearranged in complete $\mathrm{SO}(25)$ multiplets, as was just demonstrated for the first massive level. Lorentz invariance of the spectrum is guaranteed, because the Lorentz algebra is realized on the Hilbert space of transverse oscillators.

The five lowest mass levels of the oriented open bosonic string [15]

Level	$\alpha'(m)^2$	States and their $SO(24)$ representation contents	Little group	Representation contents with respect to the little group
0	-1	$\|0\rangle$ $\bullet$ (1)	$SO(1,24)$	$\bullet$ (1)
1	0	$\alpha^i_{-1}\|0\rangle$ $\square$ (24)	$SO(24)$	$\square$ (24)
2	+1	$\alpha^i_{-2}\|0\rangle$: $\square$ (24) $\alpha^i_{-1}\alpha^j_{-1}\|0\rangle$: $\square\square + \bullet$, (299) + (1)	$SO(25)$	$\square\square$ (324)
3	+2	$\alpha^i_{-3}\|0\rangle$: $\square$ (24) $\alpha^i_{-2}\alpha^j_{-1}\|0\rangle$: $\begin{smallmatrix}\square\\\square\end{smallmatrix} + \square\square + \bullet$, (276) + (299) + (1) $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\|0\rangle$: $\square\square\square + \square$, (2576) + (24)	$SO(25)$	$\square\square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}$ (2900) + (300)
4	+3	$\alpha^i_{-4}\|0\rangle$: $\square$ (24) $\alpha^i_{-3}\alpha^j_{-1}\|0\rangle$: $\begin{smallmatrix}\square\\\square\end{smallmatrix} + \square\square + \bullet$, (276) + (299) + (1) $\alpha^i_{-2}\alpha^j_{-2}\|0\rangle$: $\square\square + \bullet$, (299) + (1) $\alpha^i_{-2}\alpha^j_{-1}\alpha^k_{-1}\|0\rangle$: $2\times\square + \square\square\square + \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$, $2\times(24)$ + (2576) + (4576) $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\alpha^l_{-1}\|0\rangle$: $\square\square\square\square + \square\square + \bullet$, (17250) + (299) + (1)	$SO(25)$	$\square\square\square\square + \begin{smallmatrix}\square\square\\\square\end{smallmatrix} + \square\square + \bullet$ (20150) + (5175) + (324) + (1)

Analysis of the spectrum for the closed string

For the closed string, there are two sets of modes (left-movers and right-movers), and the level-matching condition must be taken into account. The spectrum is easily deduced from that of the open string, since closed string states are tensor products of left-movers and right-movers, each of which has the same structure as open-string states.

The mass of states in the closed-string spectrum is given by

- $N = 0$, this is a ground state $|0;k\rangle$ which has $M^2 = -\frac{4}{\alpha'}$. This is again a **tachyon** state.
- $N = 1$, the states are

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}\,|0;k\rangle\,, \qquad M^2 = 0 \tag{264}$$

corresponding to the tensor product of two massless vectors, one left-moving and one right-moving. These states form a $\mathbf{24}\otimes\mathbf{24}$ representation of $\mathrm{SO}(24)$. These decompose into three irreducible representations:

$$\begin{array}{ccccc} \texttt{traceless symmetric} & \oplus & \texttt{anti-symmetric} & \oplus & \texttt{singlet}\ (=\texttt{trace}) \\ G_{\mu\nu}(X) & & B_{\mu\nu}(X) & & \Phi(X) \\ \frac{1}{2}(D-1)(D-2)-1 & & \frac{1}{2}(D-2)(D-3) & & 1 \end{array} \tag{265}$$

These three massless fields are common to all string theories. The second part, $B_{\mu\nu}$ is an anti-symmetric tensor field which is usually called the anti-symmetric tensor field.

It also goes by the names of the "Kalb-Ramond field" or the "two-form". We don't usually encounter a two-form in QFT or GR. In fact, in four dimensions a two-form gauge field can be rewritten as a massless scalar. Actually, this two-form has the right properties to be a dark matter candidate, the **axion**, though not one that is easily seen at the LHC or via direct detection [7].

$$\begin{array}{ccccc} \texttt{traceless symmetric} & \oplus & \texttt{anti-symmetric} & \oplus & \texttt{singlet}\ (=\texttt{trace}) \\ G_{\mu\nu}(X) & & B_{\mu\nu}(X) & & \Phi(X) \\ \frac{1}{2}(D-1)(D-2)-1=299 & & \frac{1}{2}(D-2)(D-3)=276 & & 1 \end{array}$$

The scalar field Φ, the trace term, is called the **dilaton**. The first part $G_{\mu\nu}$ is the most interesting. The particle in the symmetric traceless representation of $\mathrm{SO}(24)$ is a massless spin-2 particle. However, there are general arguments, due originally to Feynman and Weinberg, that **any theory of interacting massless spin two particles must be equivalent to general relativity** [8]. Therefore the **graviton** is described by the field $G_{\mu\nu}$.

The three lowest mass levels of the oriented closed bosonic string [15]

Level	$\alpha'(m)^2$	States and their $SO(24)$ representation contents	Little group	Representation contents with respect to the little group
0	-4	$\vert 0\rangle$ $\bullet$ (1)	$SO(1,24)$	$\bullet$ (1)
1	0	$\alpha^i_{-1}\overline{\alpha}^j_{-1}\vert 0\rangle$ $\yng(1) \times \yng(1)$ (24) (24)	$SO(24)$	$\yng(2) + \yng(1,1) + \bullet$ (299) (276) (1)
2	+4	$\alpha^i_{-2}\overline{\alpha}^j_{-2}\vert 0\rangle$ $\yng(1) \times \yng(1)$ (24) (24) $\alpha^i_{-1}\alpha^j_{-1}\overline{\alpha}^k_{-1}\overline{\alpha}^l_{-1}\vert 0\rangle$ $\left(\yng(2) + \bullet\right) \times \left(\yng(2) + \bullet\right)$ $(299) + (1)$ $(299) + (1)$ $\alpha^i_{-2}\overline{\alpha}^j_{-1}\overline{\alpha}^k_{-1}\vert 0\rangle$ $\yng(1) \times \left(\yng(2) + \bullet\right)$ (24) $(299) + (1)$ $\alpha^i_{-1}\alpha^j_{-1}\overline{\alpha}^k_{-2}\vert 0\rangle$ $\left(\yng(2) + \bullet\right) \times \yng(1)$ $(299) + (1)$ (24)	$SO(25)$	$\yng(2) \times \yng(2) = \yng(4) + \yng(2,2)$ (324) (324) (20150) (32175) $+ \yng(3,1) + \yng(2) + \yng(1,1) + \bullet$ (52026) (324) (300) (1)

4 Homework Problems

PROBLEM 2.1

Consider the following classical trajectory of an open string

$$X^0 = B\tau, \quad X^1 = B\cos\tau\cos\sigma, \quad X^2 = B\sin\tau\cos\sigma, \quad X^i = 0, \quad i > 2, \tag{266}$$

and assume the conformal gauge condition.

(i) Obviously, this configuration satisfies wave equation

$$\Box X^\mu = 0 \tag{267}$$

At the same time, since

$$X'^\mu = (0, -B\cos\tau\sin\sigma, -B\sin\tau\sin\sigma, 0, \ldots, 0) \tag{268}$$

we have

$$X'^\mu\big|_{\sigma=0} = X'^\mu\big|_{\sigma=\pi} = 0 \tag{269}$$

Neumann boundary conditions are satisfied too.

Therefore the configuration (266) describes a solution to the equations of motion for the field X^μ corresponding to an open string with Neumann boundary conditions.

In static gauge, the velocity of some point of the string is defined as

$$v^i \equiv \frac{dX^i}{dX^0} = \frac{1}{B}\frac{dX^i}{d\tau} = \frac{1}{B}\big(-B\sin\tau\cos\sigma,\ B\cos\tau\cos\sigma,\ 0,\ \ldots,\ 0\big)$$
$$= \big(-\sin\tau\cos\sigma,\ \cos\tau\cos\sigma,\ 0,\ \ldots,\ 0\big) \tag{270}$$

Obviously,

$$v \equiv |v^i| = \sqrt{\cos^2\sigma} \qquad \Longrightarrow \qquad v|_{\sigma=0} = v|_{\sigma=\pi} = 1 \quad (\text{natural unit } c=1) \tag{271}$$

Therefore the endpoints of this string are indeed moving with the speed of light.

(ii) The momentum of the string is given in Eq. (82)

$$P^\mu = \int_0^\pi d\sigma P_0^\mu = T\int_0^\pi d\sigma \dot{X}^\mu \tag{272}$$

So we obtain the energy of the string

$$E = P^0 = T\int_0^\pi d\sigma \dot{X}^0 = \pi T B = \frac{B}{2\alpha'} \tag{273}$$

The conserved angular momentum tensor is given in Eq. (85)

$$J^{\mu\nu} = \int_0^\pi d\sigma J_0^{\mu\nu} = T\int_0^\pi d\sigma\big(X^\mu\dot{X}^\nu - X^\nu\dot{X}^\mu\big) \tag{274}$$

Studying the configuration of the considered string, we obtain total angular momentum

$$J = \int_0^\pi d\sigma J_0^{12} = T\int_0^\pi d\sigma\big(X^1\dot{X}^2 - X^2\dot{X}^1\big) = \frac{\pi}{2}TB^2 = \frac{B^2}{4\alpha'} \tag{275}$$

Obviously,

$$J = \alpha' E^2 \tag{276}$$

(iii) In conformal gauge $h_{ab} = \eta_{ab}$, we have

$$\begin{aligned} 0 &= T_{ab} \equiv -\frac{T}{2}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{ab}} \\ &= \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\eta_{ab}\,\eta^{cd}\,\partial_c X^\mu \partial_d X_\mu \end{aligned} \tag{277}$$

$\Longrightarrow$

$$T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0 \qquad \text{and} \qquad T_{01} = T_{10} = \dot{X} \cdot X' = 0\,. \tag{278}$$

It is easy to verify that these constraints are evidently satisfied by the considered open string configuration (266).

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